SUNSHINE COAST
66th International Mathematical Olympiad 2025

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SHORTLIST
with solutions

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Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO2025 thank the following 59 countries for contributing 235 problem proposals.

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66th International Mathematical Olympiad

Sunshine Coast QLD, Australia, 10th–20th July 2025

SUNSHINE COAST

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SHORTLISTED PROBLEMS

WITH SOLUTIONS

Problems

Algebra

A1. *Quadratic solitaire* is a single-player game. To start the game, the player chooses two distinct nonzero integers a and b and writes the equation $x^2 + ax + b = 0$ on a blackboard. On each turn, if the equation currently written on the blackboard has two distinct nonzero integer solutions $x = u$ and $x = v$, then the player erases the equation and replaces it by one of the equations $x^2 + ux + v = 0$ or $x^2 + vx + u = 0$ of their choosing. Otherwise, the game ends.
Determine all initial choices of a and b such that quadratic solitaire can be played forever.

(Czech Republic)

A2. The *sunshine cost* of a sequence $a_1, a_2, \ldots, a_{100}$ of integers is the largest possible value of

$$|(a_1 + a_2 + \cdots + a_i) - a_j|$$

as i and j vary over all integers $1, 2, \ldots, 100$.
Determine the smallest possible sunshine cost over all sequences $a_1, a_2, \ldots, a_{100}$ of pairwise distinct integers.

(China)

A3. Coral and Joey are playing the *inekoalaty game*, a two-player game whose rules depend on a positive real number λ which is known to both players. On the n^{th} turn of the game, the following happens:

- If n is odd, Coral chooses a nonnegative real number x_n such that

$$x_1 + x_2 + \cdots + x_n \leqslant \lambda n.$$

- If n is even, Joey chooses a nonnegative real number x_n such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leqslant n.$$

All chosen numbers are known to both players. If a player cannot make a move, the game ends and the other player wins.
Determine all values of λ for which Coral has a winning strategy and all those for which Joey has a winning strategy.

(Italy)

A4. Let $\mathbb{Z}_{\geqslant 0}$ be the set of all nonnegative integers. Let $f\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ be an unbounded function such that, if m and n are nonnegative integers satisfying

$$f(m+n) = \max\big\{f(0), f(1), \ldots, f(m+n)\big\},$$

then

$$f(m+n) = f(m) + f(n).$$

Prove that there exist positive integers A, B, C and D such that for all nonnegative integers n, $f(An + B) = Cn + D$.

We say that f is unbounded *if for each nonnegative integer N, there exists some nonnegative integer n such that $f(n) \geqslant N$.*

(Mongolia)

A5. Determine all integers $n \geqslant 2$ such that the polynomial

$$P(x, y, z) = x^n + y^n + z^n - x^{n-2}yz - xy^{n-2}z - xyz^{n-2}$$

can be written as $P(x, y, z) = Q(x, y, z)R(x, y, z)$, where $Q(x, y, z)$ and $R(x, y, z)$ are nonconstant polynomials with integer coefficients.

(Thailand)

A6. Let S be a set of positive integers, possibly infinite, such that no positive integer greater than 1 divides all elements of S. Determine all non-periodic infinite sequences $a_1, a_2, a_3, \ldots$ of positive integers such that, for all positive integers n,

- $a_n \leqslant |a_{n+\ell} - \ell|$ for all ℓ in S, and
- $a_n = |a_{n+\ell} - \ell|$ for at least one ℓ in S.

We say that an infinite sequence $a_1, a_2, a_3, \ldots$ is periodic *if there exists a positive integer t such that $a_n = a_{n+t}$ for all positive integers n.*

(Singapore)

A7. For each integer $k \geqslant 3$, prove that there exists a unique tuple $(x_1, x_2, \ldots, x_k)$ of positive real numbers such that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k$ and

$$\left\lfloor nx_1 + \frac{1}{2} \right\rfloor + \left\lfloor nx_2 + \frac{1}{2} \right\rfloor + \cdots + \left\lfloor nx_k + \frac{1}{2} \right\rfloor = n$$

for every integer n.

Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. For example, $\lfloor -\pi \rfloor = -4$ and $\lfloor 2 \rfloor = \lfloor 2.9 \rfloor = 2$.

(Thailand)

A8. Tim and Tam play a game. To start the game, Tim writes some nonzero real numbers, not necessarily distinct, on a blackboard. In each round, the following happens:

- First, Tam chooses a polynomial $P_k(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ whose coefficients $a_k, a_{k-1}, \ldots, a_1, a_0$ are all of the numbers currently written on the blackboard in some order. (For example, if the numbers on the blackboard are $4, -3, 4$, then $k = 2$ and Tam may choose the polynomial $4x^2 - 3x + 4$ or $-3x^2 + 4x + 4$ but not $4x - 3$ or $-3x^2 - 3x + 4$.)
- Then, if the equation $P_k(x) = 0$ has no real solutions, the game is stopped. Otherwise, Tim chooses a real number r such that $P_k(r) = 0$ and writes it on the blackboard, so there is now one more number on the blackboard.

Determine whether Tim can ensure that the game is never stopped, no matter what Tam does.

(France)

Combinatorics

C1. Let $n \geqslant 3$ be an integer, and let S_n be the set of points (x, y) in the plane such that x and y are nonnegative integers and $x + y < n$. A line in the plane is called *interesting* if it is not parallel to the x-axis, the y-axis, or the line $x + y = 0$.
Determine all nonnegative integers k such that there exist n lines satisfying both of the following:

- the union of the n lines contains every point of S_n, and
- exactly k of the lines are interesting.

(U.S.A.)

C2. There is a row of n paddocks, labelled from left to right with the integers 1 to n, where $n \geqslant 3$. Skippy the Kangaroo grazes in the paddocks according to the following rule:

> If Skippy is grazing in paddock k, then she makes a sequence of k hops from a paddock to an adjacent paddock. The first of the k hops is always to the right, unless Skippy is in paddock n, in which case it is to the left. Each of the following $k-1$ hops is in the same direction as the previous hop, unless Skippy is in paddock 1 or n. Skippy grazes in the paddock that she is in after the k^{th} hop.

For example, if $n = 8$ and Skippy is grazing in paddock 3, then the next four paddocks she grazes in are 6, 4, 8 and 2, in this order.
Skippy continues grazing in this way indefinitely. A paddock is *overgrazed* if Skippy eventually grazes in it, regardless of the paddock that she starts in. Determine all $n \geqslant 3$ for which there is exactly one overgrazed paddock.

(Colombia)

C3. There are 2025 white balls and 2025 black balls arranged in a row. A *balanced segment* is a contiguous nonempty segment of balls that contains the same number of white balls as black balls. A *balanced removal* is the operation of selecting a balanced segment S, removing all the balls in S, and shifting left the balls that were to the right of S so that the remaining balls form a contiguous row.
Determine the smallest positive integer N satisfying the following property: for every configuration of balls and for every integer k satisfying $0 \leqslant k \leqslant 2025$, there exists a sequence of at most N balanced removals after which there are exactly $2k$ remaining balls.

(Islamic Republic of Iran)

C4. Bluey is placing lollies on a 1000×1000 grid. Initially, every cell of the grid is empty. At each step, Bluey chooses a row or column in which the total number of lollies is a multiple of 3, chooses an empty cell in this row or column, and places a lolly in it. If there is no such row or column, Bluey stops.
Determine the maximum number of lollies that Bluey can place.

(China)

C5. Let n be a positive integer, and let $(a_1, a_2, \ldots, a_n)$ be any n-tuple of distinct integers. An *inversion* is a pair (i, j) of integers satisfying $1 \leqslant i < j \leqslant n$ and $a_i > a_j$. An inversion (i, j) is called *odd* if $j - i$ is odd. Let A be the number of all inversions, and let B be the number of odd inversions.

(a) Prove that $A \leqslant 5B$ for all n and all n-tuples $(a_1, a_2, \ldots, a_n)$.

(b) Prove that there exist an n and an n-tuple $(a_1, a_2, \ldots, a_n)$ for which $A > 4.99B$.

(Germany)

C6. On a 45×45 square grid there is an echidna. From any cell, the echidna can move to any other cell that shares a side. The echidna makes a sequence of 2024 moves, after which it has visited each cell exactly once. Prove that it is possible to write the numbers from 1 to 2025, one in each cell, in such a way that:

- For any pair of adjacent cells in the same row, the cell containing the larger number was visited earlier.
- For any pair of adjacent cells in the same column, the cell containing the larger number was visited later.

(Spain)

C7. Let P be a regular 100-gon. An *Aussie triangulation* is formed by dividing P into a set T of non-overlapping triangles such that:

- there are exactly 2025 different points, including the 100 vertices of P, that are a vertex of at least one triangle in T, and
- no three of these 2025 points are collinear.

A *quokkalateral* is a convex quadrilateral that consists of two triangles in T sharing a common side. Two quokkalaterals may share a triangle in T.
Determine the minimum number of quokkalaterals among all Aussie triangulations.

(China)

C8. Consider a 2025×2025 grid of unit squares. We place on the grid some number of rectangular tiles, possibly of different sizes, such that each side of every tile lies on a grid line and every unit square is covered by at most one tile.
Determine the minimum number of tiles we need to place so that each row and each column has exactly one unit square that is not covered by any tile.

(Singapore)

Geometry

G1. Let ABC be a triangle such that $\angle A$ is obtuse. Let D and E be points on sides AB and AC, respectively, such that $BDEC$ is cyclic. Suppose the circumcircle of triangle ADE intersects side BC at two points X and Y, where B, X, Y and C lie in that order. Let the circumcircles of triangles BDX and CEY be Ω and Γ, respectively. Let P be a point such that PD is tangent to Ω and PE is tangent to Γ. Let Q be a point such that QB is tangent to Ω and QC is tangent to Γ.
Prove that points A, P and Q are collinear.

(Czech Republic)

G2. Let ABC be an acute-angled triangle. Let I_B and I_C be the excentres of triangle ABC opposite B and C, respectively. Let E be a point on line BA such that $\angle EI_BB = 90^\circ$. Let F be a point on line CA such that $\angle CI_CF = 90^\circ$. Let the circle with centre I_B and radius I_BE intersect the segment I_BB at X. Let the circle with centre I_C and radius I_CF intersect the segment I_CC at Y.
Prove that BY is perpendicular to CX.

The excentre *of a triangle ABC opposite vertex B is the centre of the circle that is tangent to line segment AC, to ray BA beyond A, and to ray BC beyond C. The excentre opposite C is similarly defined.*

(Islamic Republic of Iran)

G3. Let ABC be an acute-angled triangle with $AB < AC$. Let Γ be the circumcircle of triangle ABC. The perpendicular from B to AC intersects Γ again at $D \neq B$. Let Q be the point on Γ such that $Q \neq B$ and $AB = AQ$. Lines BQ and AC intersect at R. Line DR intersects Γ again at $S \neq D$. Segment BC intersects the circumcircle of triangle QRS at T. Lines TQ and RS intersect at X.
Prove that CX is perpendicular to AB.

(Kosovo)

G4. Let Ω and Γ be circles centred at M and N, respectively, such that the radius of Ω is smaller than the radius of Γ. Suppose circles Ω and Γ intersect at two distinct points A and B. Line MN intersects Ω at C and Γ at D, such that points C, M, N and D lie on the line in that order. Let P be the circumcentre of triangle ACD. Line AP intersects Ω again at $E \neq A$. Line AP intersects Γ again at $F \neq A$. Let H be the orthocentre of triangle PMN.
Prove that the line passing through H parallel to AP is tangent to the circumcircle of triangle BEF.

(Vietnam)

G5. Let $ABCD$ be a convex quadrilateral with $\angle ADC > 90^\circ$. Suppose that points X and Y lie on diagonal AC such that

$$\angle ADX = \angle YDC = 90^\circ, \quad \angle CAD = \angle CBX, \quad \text{and} \quad \angle DCA = \angle YBA.$$

Let O be the circumcentre of triangle BXY. Suppose point $R \neq O$ lies on the perpendicular bisector of AC such that lines OR and AC are parallel.
Prove that line RO bisects angle $\angle DRB$.

(Romania)

G6. Let ABC be a triangle with $\angle B > \angle A > \angle C$. Let Ω be the circumcircle of triangle ABC. The lines tangent to Ω at points B and C intersect at P. Lines BP and AC intersect at E. Lines CP and AB intersect at F. Let D be the reflection of point A in line EF. The circumcircles of triangles DEB and DFC intersect again at $Q \neq D$. Lines DP and BC intersect at Z.
Prove that points A, Z and Q are collinear.

(India)

G7. Let $ABCD$ be a cyclic quadrilateral with circumcircle Ω such that $CD > BD > BC$. The tangents to Ω at B and C meet at a point S. A line ℓ is called *coastal* if:

- ℓ does not pass through A,
- ℓ intersects ray SD at a point X_ℓ beyond D, and
- there is a circle tangent to line ℓ, line $X_\ell A$, segment SB and segment SC.

Prove that there is a circle that is tangent to Ω and all coastal lines.

(Islamic Republic of Iran)

Number Theory

N1. Let n and k be positive integers such that $n < k < 2n$. Suppose $a_1, a_2, \ldots, a_k$ are positive integers such that $2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}$ is divisible by $2^n - 1$.
Show that at least three of $a_1, a_2, \ldots, a_k$ have the same remainder when divided by n.

(Mongolia)

N2. In an $n \times n$ board, for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, the cell in the i^{th} row from the bottom and j^{th} column from the left contains $\gcd(i, j)$ leaves. A koala travels from the bottom left corner to the top right corner. At each step, the koala moves up or to the right by a single cell. The koala eats the leaves in each cell it visits, including the bottom left and top right cells.
Determine the maximum number of leaves that the koala can eat.

Here $\gcd(x, y)$ *denotes the greatest common divisor of integers* x *and* y.

(India)

N3. An infinite sequence $a_1, a_2, \ldots$ consists of positive integers, each of which has at least four positive divisors. For each $n \geqslant 1$, a_{n+1} is the sum of the three largest proper divisors of a_n.
Determine all possible values of a_1.

A proper divisor *of a positive integer* N *is a divisor of* N *other than* N.

(Lithuania)

N4. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Determine all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}$ such that for every $n \in \mathbb{Z}_{>0}$, and every positive divisor d of n, there exists a positive divisor e of n such that n divides $d + f(e)$.

(Croatia)

N5. The sequence of triples of positive integers $(a_0, b_0, c_0), (a_1, b_1, c_1), \ldots, (a_{2025}, b_{2025}, c_{2025})$ satisfies

$$(a_{i+1}, b_{i+1}, c_{i+1}) = \big(a_i + \gcd(b_i, c_i), b_i + \gcd(c_i, a_i), c_i + \gcd(a_i, b_i)\big)$$

for $0 \leqslant i \leqslant 2024$.
Determine the smallest possible value of a_{2025}.

Here $\gcd(x, y)$ *denotes the greatest common divisor of integers* x *and* y.

(India)

N6. For positive integers n and k, let $f_k(n)$ denote the remainder when n is divided by $2^k - 1$. A positive integer n is *grouse* if

$$f_1(n) \leqslant f_2(n) \leqslant f_3(n) \leqslant f_4(n) \leqslant \cdots .$$

(a) Prove that there are infinitely many grouse numbers.

(b) Prove that there exists a positive integer M such that there are at most $M/2025^{2025}$ grouse numbers less than M.

(U.S.A.)

N7. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. A function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is said to be *bonza* if

$$f(a) \text{ divides } b^a - f(b)^{f(a)}$$

for all positive integers a and b.
Determine the smallest constant c such that $f(n) \leqslant cn$ for any bonza function f and any positive integer n.

(Colombia)

N8. Prove that there are finitely many prime numbers p such that

- $p - 8$ is a perfect square, and
- for all odd positive integers $k < \sqrt{p}$, the number $p - k^2$ has at most two distinct prime factors.

(Romania)

Solutions

Algebra

A1. *Quadratic solitaire* is a single-player game. To start the game, the player chooses two distinct nonzero integers a and b and writes the equation $x^2 + ax + b = 0$ on a blackboard. On each turn, if the equation currently written on the blackboard has two distinct nonzero integer solutions $x = u$ and $x = v$, then the player erases the equation and replaces it by one of the equations $x^2 + ux + v = 0$ or $x^2 + vx + u = 0$ of their choosing. Otherwise, the game ends.

Determine all initial choices of a and b such that quadratic solitaire can be played forever.

(Czech Republic)

Answer: The only pair (a, b) for which quadratic solitaire can be played forever is $(1, -2)$.

Solution 1. Let (a, b) be a pair of distinct nonzero integers such that quadratic solitaire can be played forever. We will work with a sequence of turns for which quadratic solitaire goes on forever. Consider a turn in that sequence where the equation $x^2 + sx + t = 0$ is replaced by $x^2 + ux + v = 0$.

Vieta's formula gives $t = uv$. Since u is a nonzero integer, we have $|t| = |uv| \geqslant |v|$. The value of $|v|$ cannot decrease below zero, so there can be only finitely many turns with $|u| \neq 1$. Therefore, if the game continues forever, then eventually $|s| = |u| = 1$ every turn. Vieta's formula $s = -(u + v)$ then leads to the following three cases:

- If $s = 1$ and $u = 1$, then $v = -2$. The equation $x^2 + ux + v = x^2 + x - 2 = 0$ has solutions $\{1, -2\}$, from which the player can continue playing quadratic solitaire forever.
- If $s = -1$ and $u = -1$, then $v = 2$. The equation $x^2 + ux + v = x^2 - x + 2 = 0$ does not have integer solutions, so the game ends.
- If $\{s, u\} = \{1, -1\}$, then $v = 0$, which is impossible.

In conclusion, we have shown that the only way for quadratic solitaire to continue forever is to encounter the equation $x^2 + x - 2 = 0$ at some turn.

Conversely, if $x^2 + x - 2 = 0$ is the equation on the blackboard at the start of a turn, then the equation written down on the previous turn had to be $(x - 1)(x + 2) = x^2 + x - 2 = 0$. Therefore, in order to encounter the equation $x^2 + x - 2 = 0$ during some turn, the player must have started with the same equation, implying that $(a, b) = (1, -2)$.

Solution 2. We provide an alternative solution using a different monovariant. As in Solution 1, consider a turn where the equation $x^2 + sx + t = 0$ is replaced by $x^2 + ux + v = 0$.

Claim 1. $uv \neq -1$.

Proof. If $uv = -1$, then we would have $\{u, v\} = \{1, -1\}$ because u and v are integers. In the next turn, the new equations to consider would be $x^2 + x - 1 = 0$ and $x^2 - x + 1 = 0$. Neither of these has integer solutions, so the game would have ended. □

Claim 2. $s^2 + t^2 \geqslant u^2 + v^2$ and equality holds if and only if $uv = -2$.

Proof. Since u and v are the two distinct solutions of $x^2 + sx + t = 0$, Vieta's formulas give $s = -(u + v)$ and $t = uv$. So, in order to prove that $s^2 + t^2 \geqslant u^2 + v^2$, we need to show that $(u + v)^2 + u^2v^2 \geqslant u^2 + v^2$. This is equivalent to $(uv + 1)^2 \geqslant 1$, which is true since $uv + 1$ is a nonzero integer by the previous claim. Equality holds if and only if $uv = 0, -2$. The former is impossible as u, v are nonzero integers. □

Since the value of u^2+v^2 cannot decrease below zero, there can be only finitely many turns with $uv \neq -2$. It remains to discuss what happens if $uv = -2$.

If $uv = -2$, then we must have $\{u, v\} = \{1, -2\}$ or $\{u, v\} = \{-1, 2\}$. From there, we get four possible equations for the next turn:

- $x^2 + x - 2 = 0$: two integer solutions.
- $x^2 - 2x + 1 = 0$: one (repeated) integer solution.
- $x^2 - x + 2 = 0$: no integer solutions.
- $x^2 + 2x - 1 = 0$: no integer solutions.

Only the first equation allows the game to continue. The solution can be completed as before.

Comment. We are aware that some problems look superficially similar to this problem (for instance EGMO 2024 Problem 1). Another example is China Girls Math Olympiad 2020 Problem 5, which reads

> Find all sequences of *real* numbers $(b_n)_{n\geqslant 1}$ and $(c_n)_{n\geqslant 1}$ such that $b_n \leqslant c_n$ for every $n \geqslant 1$, and $\{b_{n+1}, c_{n+1}\}$ are the two solutions of the equation $x^2 + b_n x + c_n = 0$ for every $n \geqslant 1$.

Despite similarities in the statements, the problems themselves are different. Proving that $x^2 + x - 2$ is the only non-trivial monic quadratic polynomial whose roots are its coefficients is also a known problem, but only contributes to a straightforward check at the end of the solution.

A2. The *sunshine cost* of a sequence $a_1, a_2, \ldots, a_{100}$ of integers is the largest possible value of

$$|(a_1 + a_2 + \cdots + a_i) - a_j|$$

as i and j vary over all integers $1, 2, \ldots, 100$.

Determine the smallest possible sunshine cost over all sequences $a_1, a_2, \ldots, a_{100}$ of pairwise distinct integers.

(China)

Answer: The smallest possible sunshine cost of such a sequence is 75.

Solution 1. We fix a sequence $a_1, a_2, \ldots, a_{100}$ of pairwise distinct integers. We will first prove that its sunshine cost C satisfies $C \geqslant 75$. Let us denote the largest and the smallest values achieved by the sequence by M and m, respectively. Let $u, v \in \{1, 2, \ldots, 100\}$ be the two indices such that $a_u = M$ and $a_v = m$. Since the integers a_i are pairwise distinct, we have $M - m \geqslant 99$. We can further assume without loss of generality that $M + m \geqslant 0$, otherwise we can instead consider the sequence $-a_1, -a_2, \ldots, -a_{100}$ which also has sunshine cost C. We will now consider several cases depending on the values of m and u.

- If $m \geqslant -49$, then by taking $i = 100$ and $j = v$, we have

  $$C \geqslant |a_1 + a_2 + \cdots + a_{100} - a_v| \geqslant a_1 + a_2 + \cdots + a_{100} - a_v,$$

  by definition of C and because $|x| \geqslant x$. Since the integers a_i are distinct, it holds that

  $$a_1 + a_2 + \cdots + a_{100} - a_v \geqslant 99m + 1 + 2 + \cdots + 99 = 99(m + 50).$$

  We conclude that $C \geqslant 99$.

- If $m \leqslant -50$ and $u = 1$, then by taking $i = 1$ and $j = v$, we get $C \geqslant |a_1 - a_v| = M - m \geqslant 99$.

- If $m \leqslant -50$ and $u \geqslant 2$, then by first considering $i = u - 1$ and $j = u$, we obtain $C \geqslant a_u - (a_1 + \cdots + a_{u-1})$, since $|x| \geqslant \pm x$. Then, by taking $i = u$ and $j = v$, we also have $C \geqslant (a_1 + \cdots + a_u) - a_v$. Hence,

  $$2C \geqslant a_u - (a_1 + \cdots + a_{u-1}) + (a_1 + \cdots + a_u) - a_v = 2a_u - a_v = 2M - m = 2(M + m) - 3m \geqslant 150,$$

  and thus $C \geqslant 75$.

In conclusion, we do indeed have $C \geqslant 75$ in all cases.

We now construct a sequence $a_1, a_2, \ldots, a_{100}$ of pairwise distinct integers whose sunshine cost is equal to 75. Consider for instance the sequence

$$25, -50, 50, -49, 49, \ldots, -26, 26, -25, 24, -24, 23, \ldots, 1, -1.$$

Notice that, in this example, $-25 \leqslant a_1 + a_2 + \cdots + a_i \leqslant 25$ for every $i = 1, 2, \ldots, 100$. This implies $|(a_1 + a_2 + \cdots + a_i) - a_j| \leqslant 75$ with equality achieved for $i = 1$ and $j = 2$. Therefore, the sunshine cost of this sequence is 75.

Another example is the sequence

$$1, -2, 3, -4, \ldots, -48, 49, -50, 50, -49, 48, \ldots, 4, -3, 2, -1.$$

It also satisfies $-25 \leqslant a_1 + a_2 + \cdots + a_i \leqslant 25$ for every $i = 1, 2, \ldots, 100$ and $|(a_1 + a_2 + \cdots + a_i) - a_j| \leqslant 75$. Equality is now achieved for $i = 50$ and $j = 51$ (or for $i = 51$ and $j = 50$).

Solution 2. We give an alternative argument to prove that the sunshine cost C of a sequence $a_1, a_2, \ldots, a_{100}$ of pairwise distinct integers satisfies $C \geqslant 75$. We use the same notation as in Solution 1.

First, consider the case where $|a_i| \leqslant 49$ for every $i \geqslant 2$. This implies that $\{a_2, a_3, \ldots, a_{100}\} = \{-49, -48, \ldots, 48, 49\}$ and $|a_1| \geqslant 50$ because the a_i are pairwise distinct. If $a_1 > 0$, then by taking $i = 1$ and j to be the (unique) index such that $a_j = -49$, we obtain $C \geqslant |a_1 - a_j| \geqslant 99$. If instead $a_1 < 0$, then by taking $i = 1$ and j to be the (unique) index such that $a_j = 49$, we similarly obtain $C \geqslant |a_1 - a_j| \geqslant 99$.

We will now assume that $|a_i| \geqslant 50$ for some index $i \geqslant 2$. We may assume without loss of generality that $a_i \geqslant 50$, since otherwise, we can consider the sequence $-a_1, -a_2, \ldots, -a_{100}$ instead. Let S and s respectively denote the largest and the smallest values of the partial sums $a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{100}$. With this notation, we have

$$C = \max\big(|s - M|, |S - m|\big).$$

Note that

$$S - s \geqslant (a_1 + a_2 + \cdots + a_i) - (a_1 + a_2 + \cdots + a_{i-1}) = a_i \geqslant 50.$$

Combining the above with the triangle inequality and $M - m \geqslant 99$, we obtain

$$2C \geqslant |s - M| + |S - m| \geqslant |M - m + S - s| = M - m + S - s \geqslant 149.$$

Since C is an integer, we conclude that $C \geqslant 75$.

Comment. The index 100 in the problem statement can be replaced by any other positive integer n. The smallest possible sunshine cost of a sequence of pairwise distinct integers is then given by

$$\begin{cases} 3k - 1 & \text{if } n = 4k - 2 \text{ or } 4k - 1, \\ 3k & \text{if } n = 4k \text{ or } 4k + 1. \end{cases}$$

For instance, when $n \in \{2700, 2701\}$, then the answer to the problem is 2025. Similar solutions will work in general, but several steps may be slightly more tedious when n is not a multiple of 4.

A3. Coral and Joey are playing the *inekoalaty game*, a two-player game whose rules depend on a positive real number λ which is known to both players. On the n^{th} turn of the game, the following happens:

- If n is odd, Coral chooses a nonnegative real number x_n such that
$$x_1 + x_2 + \cdots + x_n \leqslant \lambda n.$$
- If n is even, Joey chooses a nonnegative real number x_n such that
$$x_1^2 + x_2^2 + \cdots + x_n^2 \leqslant n.$$

All chosen numbers are known to both players. If a player cannot make a move, the game ends and the other player wins.

Determine all values of λ for which Coral has a winning strategy and all those for which Joey has a winning strategy.

(Italy)

Answer:

(a) If $\lambda < \frac{1}{\sqrt{2}}$, then Joey has a winning strategy.

(b) If $\lambda > \frac{1}{\sqrt{2}}$, then Coral has a winning strategy.

(c) When $\lambda = \frac{1}{\sqrt{2}}$, then neither player has a winning strategy.

Solution. We consider all three regimes individually.

(a) We start with the case $\lambda < \frac{1}{\sqrt{2}}$. We will prove that a winning strategy for Joey consists of choosing the largest possible number every turn. More precisely, Joey will choose
$$x_{2k} = \sqrt{2 - x_{2k-1}^2}$$
during every even turn of the inekoalaty game.

On the first turn, Coral chooses a nonnegative number $x_1 \leqslant \lambda$. Since $\lambda < \frac{1}{\sqrt{2}}$, it follows that $2 - x_1^2 \geqslant 0$ and therefore Joey can choose x_2 to be $\sqrt{2 - x_1^2}$, which ensures $x_1^2 + x_2^2 = 2$. On the third turn, Coral has to choose a nonnegative number x_3 such that $x_1 + x_2 + x_3 \leqslant 3\lambda$. Either she cannot and Joey wins, or the largest number x_3 she can choose must satisfy
$$x_3 \leqslant 3\lambda - x_1 - \sqrt{2 - x_1^2} \leqslant 3\lambda - \sqrt{2} < \lambda.$$
We used here that $x + \sqrt{2 - x^2} \geqslant \sqrt{2}$ for all $x \in [0, \sqrt{2}]$, which can be established by squaring both sides. Now, since $x_3 < \lambda$, it follows that $2 - x_3^2 \geqslant 0$ and so Joey may choose $x_4 = \sqrt{2 - x_3^2}$, which gives $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4$. By repeating the same argument, we can inductively prove that, as long as Joey keeps choosing $x_{2k} = \sqrt{2 - x_{2k-1}^2}$, then Coral either loses in the next turn or must choose a nonnegative number x_{2k+1} satisfying $x_{2k+1} < \lambda$. In other words, Joey's strategy is valid and ensures that he will always be able to make a legal move, while all numbers chosen by Coral are smaller than λ. On the other hand, Joey's strategy also implies that
$$x_1 + x_2 + \cdots + x_{2k} = \sum_{i=1}^{k} \left(x_{2i-1} + \sqrt{2 - x_{2i-1}^2} \right) \geqslant k\sqrt{2},$$

where we used $x + \sqrt{2 - x^2} \geqslant \sqrt{2}$ again. Consequently, as soon as k is large enough such that $k\sqrt{2} > \lambda(2k + 1)$ (such a k necessarily exists since $\lambda < \frac{1}{\sqrt{2}}$), then Coral does not have a legal move on turn $2k + 1$ and Joey wins.

(b) Let us next consider the case where $\lambda > \frac{1}{\sqrt{2}}$. Coral's winning strategy consists of choosing $x_n = 0$ for several turns starting from the beginning, and then choosing x_n to be "large" on a later turn. We will now make this strategy precise and prove that it works.

We first show that Coral can choose $x_n = 0$ for as many consecutive turns as she wishes. This is obviously possible on the first turn. After $2k$ turns, if Coral chose $x_n = 0$ on turns $n = 1, 3, \ldots, 2k - 1$, then

$$x_1 + x_2 + \cdots + x_{2k} = x_2 + x_4 + \cdots + x_{2k} \leqslant \sqrt{k(x_2^2 + x_4^2 + \cdots + x_{2k}^2)},$$

by the AM-QM inequality. Assuming that Joey has not lost yet, it must hold that $x_2^2 + x_4^2 + \cdots + x_{2k}^2 \leqslant 2k$, which implies $x_1 + x_2 + \cdots + x_{2k} \leqslant k\sqrt{2} < \lambda(2k + 1)$ because $\lambda > \frac{1}{\sqrt{2}}$. This means that Coral can legally choose $x_{2k+1} = 0$ on turn $2k + 1$, no matter what the numbers chosen by Joey so far are. More precisely, Coral can actually choose x_{2k+1} to be any nonnegative number satisfying

$$x_{2k+1} \leqslant \lambda(2k + 1) - k\sqrt{2} = k(2\lambda - \sqrt{2}) + \lambda.$$

Note that this upper bound on x_{2k+1} gets arbitrarily large as k increases, since $2\lambda - \sqrt{2} > 0$.

We are now ready to describe Coral's winning strategy. Let ℓ be a large enough integer such that

$$\frac{\ell\sqrt{2} + \sqrt{2\ell + 2}}{2\ell + 1} < \lambda.$$

Such an ℓ always exists since $\lambda > \frac{1}{\sqrt{2}}$. Coral's strategy is to choose $x_n = 0$ on turns $n = 1, 3, \ldots, 2\ell - 1$ and then choose

$$x_{2\ell+1} = \lambda(2\ell + 1) - \ell\sqrt{2}$$

on turn $2\ell + 1$. We already proved that these choices are always legal, no matter what Joey's decisions were. Now, observe that

$$x_1^2 + x_2^2 + \cdots + x_{2\ell+1}^2 \geqslant x_{2\ell+1}^2 = \left(\lambda(2\ell + 1) - \ell\sqrt{2}\right)^2 > 2\ell + 2,$$

by the definitions of $x_{2\ell+1}$ and ℓ. This implies that Joey has no legal move on turn $2\ell + 2$ and therefore loses the game.

(c) Finally, we consider the case where $\lambda = \frac{1}{\sqrt{2}}$. We will prove that neither player has a winning strategy in that case. The same argument as in part (a) shows that Joey can always make a legal move by choosing $x_{2k} = \sqrt{2 - x_{2k-1}^2}$ on turn $2k$ and is therefore sure to never lose. Similarly, by the argument in part (b), Coral can always make a legal move by choosing $x_{2k+1} = 0$ on turn $2k + 1$ and is therefore sure to never lose.

A4. Let $\mathbb{Z}_{\geqslant 0}$ be the set of all nonnegative integers. Let $f\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ be an unbounded function such that, if m and n are nonnegative integers satisfying

$$f(m+n) = \max\big\{f(0), f(1), \ldots, f(m+n)\big\},$$

then

$$f(m+n) = f(m) + f(n).$$

Prove that there exist positive integers A, B, C and D such that for all nonnegative integers n, $f(An+B) = Cn + D$.

We say that f is unbounded *if for each nonnegative integer N, there exists some nonnegative integer n such that $f(n) \geqslant N$.*

(Mongolia)

Solution 1. Let f be an unbounded function that satisfies the condition in the problem statement. An integer $a \geqslant 0$ is called *big* if $f(b) \leqslant f(a)$ for all $b \leqslant a$. When a is big, we have $f(a) = f(b) + f(a-b)$ for all $b \leqslant a$, by the condition on f. In particular, since 0 is big, we have $f(0) = f(0) + f(0)$, and hence $f(0) = 0$. We will also say that an integer $a \geqslant 0$ is *huge* if $f(b) < f(a)$ for all $b < a$. In particular, 0 is huge and huge integers are big.

Claim 1.1. There are infinitely many huge integers.

Proof. If there were only finitely many huge integers, then the largest one would be an upper bound on f. □

Claim 1.2. $f(n) \geqslant 1$ for every $n \geqslant 1$.

Proof. By Claim 1.1, for every $n \geqslant 1$, there exists a huge integer $a > n$. Hence, $f(n) = f(a) - f(a-n) > 0$. □

Claim 1.3. All big integers are huge.

Proof. Recall that 0 is both big and huge. If $a \geqslant 1$ were big but not huge, then there would exist an integer b with $0 \leqslant b < a$ such that $f(b) = f(a)$. Since a is big, $f(a) = f(b) + f(a-b)$ and so $f(a-b) = 0$. This contradicts Claim 1.2 because $a - b \geqslant 1$. □

If a is huge, we denote by a^+ the next huge integer greater than a.

Claim 1.4. If a is huge and $n \geqslant 0$ is an integer with $n < a^+$ and $n \neq a$, then $f(n) < f(a)$.

Proof. As a is huge, the conclusion immediately follows if $n < a$. Now take m to be the least integer $m > a$ such that $f(m) \geqslant f(a)$. The minimality assumption on m implies that m is big, and thus huge by Claim 1.3. Thus $m = a^+$ and the claim follows. □

Claim 1.5. If a is huge and $m \geqslant 1$ is an integer with $m \leqslant a^+$ and $m \neq a^+ - a$, then $f(m) > f(a^+ - a)$.

Proof. We apply Claim 1.4 with $n = a^+ - m < a^+$ and obtain $f(a) > f(a^+ - m)$. This gives

$$f(m) = f(a^+) - f(a^+ - m) > f(a^+) - f(a) = f(a^+ - a).$$ □

Let C be the smallest positive value f takes, and let A be the smallest positive integer such that $f(A) = C$. Let B be a huge integer with $B \geqslant A$. Considering Claim 1.5 with $a = B$ and $m = A$, we have $1 \leqslant A \leqslant B^+$, but $f(A) > f(B^+ - B)$ is false by minimality of $f(A)$, so $A = B^+ - B$. Hence $B^+ = A + B$ is huge, and $f(A+B) = f(A) + f(B) = C + f(B)$.

We now prove by induction that, for any nonnegative integer n, $An + B$ is huge, and $f(An+B) = Cn + f(B)$; the $n = 0$ case is clear and $n = 1$ is proved above. Again considering Claim 1.5, with $m = A$ and $a = An + B$, we have $1 \leqslant A \leqslant (An+B)^+$, but $f(A) > f((An + B)^+ - (An+B))$ is false, so $A = (An+B)^+ - (An+B)$, hence $(An+B)^+ = A(n+1) + B$. So $A(n+1) + B$ is huge and $f(A(n+1)+B) = f(An+B) + f(A)$, which by induction is equal to $C(n+1) + f(B)$. The conclusion then follows after taking $D = f(B)$.

Solution 2. Proceed as in Solution 1, noting that $f(0) = 0$, up to the end of Claim 1.1. We consider $A \geqslant 1$ to be any integer such that $f(A)$ is the smallest positive value of f. We use the same terminology as in Solution 1.

Claim 2.1. If a is big and $a \geqslant A$, then $a - A$ is also big.

Proof. For any $n \leqslant a - A$, using the fact that a is big and $f(A) \leqslant f(a - n)$, we have

$$f(a - A) = f(a) - f(A) \geqslant f(a) - f(a - n) = f(n). \qquad \square$$

Claim 2.2. There exists an integer B such that $1 \leqslant B \leqslant A$ and $nA + B$ is big for any $n \geqslant 1$.

Proof. By Claim 1.1 of Solution 1, there are infinitely many big integers. By the infinite pigeonhole principle, some residue class modulo A contains infinitely many big numbers. In other words, there exists $1 \leqslant B \leqslant A$ such that there are infinitely many big numbers of the form $nA + B$. Claim 2.1 says that if $nA + B$ is big and $n \geqslant 1$, then $(n - 1)A + B$ is also big. So, all integers of the form $nA + B$ are indeed big for $n \geqslant 1$. $\square$

Claim 2.2 implies that, for every $n \geqslant 1$,

$$f(nA + B) = f((n - 1)A + B) + f(A) = \cdots = nf(A) + f(B).$$

Setting $C = f(A)$ and $D = f(B)$, we get the desired conclusion.

A5. Determine all integers $n \geqslant 2$ such that the polynomial

$$P(x, y, z) = x^n + y^n + z^n - x^{n-2}yz - xy^{n-2}z - xyz^{n-2}$$

can be written as $P(x, y, z) = Q(x, y, z)R(x, y, z)$, where $Q(x, y, z)$ and $R(x, y, z)$ are nonconstant polynomials with integer coefficients.

(Thailand)

Answer: The only n for which the polynomial can be factored is $n = 3$.

Solution 1. For a polynomial S with integer coefficients, if it cannot be factored as AB for nonconstant polynomials A and B with integer coefficients, we say that S is *irreducible*.

Note that homogeneous polynomials can only be factored into homogeneous polynomials, so if $P(x, y, z) = Q(x, y, z)R(x, y, z)$, where $Q(x, y, z)$ and $R(x, y, z)$ are nonconstant, this would mean that $Q(x, y, z)$ and $R(x, y, z)$ both have degree at most $n-1$ in each of x, y, z. Also, note that viewed as a polynomial in x, the leading coefficient of P is 1. This in particular means that viewed as polynomials in x, the leading coefficients of Q and R must also be ± 1, so if we plug in $y = \pm z$ or particular values of y and z, the resulting Q and R remain nonconstant in x.

The strategy is to consider the polynomial $P(x, y, -y)$ and divide into two cases according to the parity of n.

Case 1: n is even.

For $n = 2k$,

$$P(x, y, -y) = x^{2k} + 2y^{2k} + x^{2k-2}y^2.$$

If we plug in $y = 1$, this is

$$x^{2k} + x^{2k-2} + 2.$$

Lemma. The polynomial $S(x) = x^{2k} + x^{2k-2} + 2$ cannot be factored into $A(x)B(x)$ for polynomials $A(x), B(x)$ with integer coefficients of degree less than $2k$.

Proof. For $k = 1$, we have that $S(x) = x^2 + 3$, which is clearly irreducible. We now assume that $k \geqslant 2$. Suppose for the sake of contradiction that a factorisation $S(x) = A(x)B(x)$ exists. We may assume that $A(x)$ and $B(x)$ both have leading coefficient 1.

Note that $A(x)B(x) = S(x) \equiv x^{2k-2}(x^2 + 1) \pmod 2$. Let $\overline{A}(x)$ and $\overline{B}(x)$ denote the reductions of $A(x)$ and $B(x)$ modulo 2. Since the product of the constant terms of $A(x)$ and $B(x)$ is 2, the constant terms of $A(x)$ and $B(x)$ cannot both be even, so the constant terms of $\overline{A}(x)$ and $\overline{B}(x)$ cannot both be 0. Thus, one of them is not a multiple of x. Without loss of generality, suppose it is $\overline{A}(x)$. Because $\overline{A}(x)\overline{B}(x) = x^{2k-2}(x^2 + 1)$, and polynomials factor uniquely, this means $\overline{A}(x)$ divides $x^2 + 1$, which in particular means it has degree either 1 or 2.

Since $A(x)$ and $B(x)$ have leading coefficient 1, we know that $\overline{A}(x)$ and $\overline{B}(x)$ have the same degrees as $A(x)$ and $B(x)$, respectively. Since $S(x)$ clearly does not have an integer root (or, indeed, even a real root), we know that $A(x)$ cannot have degree 1. Thus $A(x)$ has degree 2.

Since $A(x)$ has leading coefficient 1 and final coefficient ± 1 (since the final coefficient is a factor of 2), it must be $x^2 + 2mx \pm 1$. If $m = 0$ then $x^2 + 2mx \pm 1$ does not divide $S(x)$, because $x^2 \pm 1$ has i or 1 as a root, and $S(x)$ does not. For any other $m \in \mathbb{Z}$, $x^2 + 2mx \pm 1$ has a real solution, which $S(x)$ does not have. □

We conclude that the polynomial $x^{2k} + 2y^{2k} + x^{2k-2}y^2$ is irreducible, which implies that $P(x, y, z)$ is irreducible too.

Comment. The above lemma can be viewed as a consequence of a natural generalisation of Eisenstein's criterion, and the proof for it is inspired by the proof of Eisenstein's criterion.

Case 2: n is odd.

For $n = 2k + 1$, plug in

$$P(x, y, -y) = x^{2k+1} + x^{2k-1}y^2 + 2y^{2k}x = x(x^{2k} + x^{2k-2}y^2 + 2y^{2k}).$$

Applying the lemma above, $x^{2k} + x^{2k-2}y^2 + 2y^{2k}$ is irreducible, so if $P(x, y, z)$ factors, then one of the factors (which are homogeneous) has degree $2k$, and the other has degree 1. Also, both factors have leading coefficient 1 in x, so the degree 1 term has the form $x + ay + az$. By applying symmetric arguments to $P(x, -x, z)$ and $P(-z, y, z)$, we see that the coefficients of x, y, z are all the same, so $a = 1$. Thus, the factor is $x + y + z$. In other words, if $P(x, y, z)$ factors into $Q(x, y, z)R(x, y, z)$, then one of the factors must be $x+y+z$. In this case, plugging in $(x, y, z) = (2, -1, -1)$, we get that $x + y + z = 0$, so

$$0 = P(2, -1, -1) = 2^{2k+1} - 1 - 1 - 2^{2k-1} - 2 - 2 = 2^{2k+1} - 2^{2k-1} - 6 = 2^{2k-1} \cdot 3 - 6,$$

implying $k = 1$ and $n = 3$. For $n = 3$, the polynomial can be factored as

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

In conclusion, the only n for which the polynomial can be factored is $n = 3$.

Solution 2. Note that for $n = 2$, $P(x, y, z) = x^2 + y^2 + z^2 - xy - yz - zx$. Clearly $P(x, 1, 0) = x^2 - x + 1$ is irreducible over $\mathbb{Z}$, as it does not have an integer root. Thus, $P(x, y, z)$ is irreducible over $\mathbb{Z}$. For $n = 3$, the polynomial can be factored as $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$. We now assume that $n > 3$.

Observe that

$$\begin{aligned}
P(x, y, y) &= x^n + 2y^n - x^{n-2}y^2 - 2xy^{n-1} \\
&= x^n - x^{n-1}y + x^{n-1}y - x^{n-2}y^2 - 2xy^{n-1} + 2y^n \\
&= (x - y)(x^{n-1} + x^{n-2}y - 2y^{n-1}) \\
&= (x - y)(x^{n-1} - x^{n-2}y + 2x^{n-2}y - 2y^{n-1}) \\
&= (x - y)^2(x^{n-2} + 2(x^{n-3}y + x^{n-4}y^2 + \cdots + y^{n-2})) \\
&= (x - y)^2 \left(x^{n-2} + 2 \sum_{k=1}^{n-2} x^{n-2-k}y^k \right).
\end{aligned}$$

Plugging in $y = 1$ into the factor $x^{n-2} + 2\sum_{k=1}^{n-2} x^{n-2-k}y^k$ gives $x^{n-2} + 2\sum_{k=1}^{n-2} x^{n-2-k}$, which is irreducible by Eisenstein's criterion. Thus, if $P(x, y, z)$ factors, it must have an irreducible factor of degree at least $n - 2$.

Case 1: $P(x, y, z)$ has an irreducible factor of degree 1.

Suppose this factor is $Q(x, y, z) = ax+by+cz$ for integers a, b, c. If the integers a, b, c are not all equal, then by noting that $P(x, y, z)$ is cyclically symmetric, that is $P(x, y, z) = P(y, z, x) = P(z, x, y)$, we see that $Q(x, y, z)$, $Q(y, z, x)$ and $Q(z, x, y)$ are all factors of $P(x, y, z)$. Note that $Q(x, y, z)$ cannot be a multiple of $Q(y, z, x)$; otherwise, if $Q(y, z, x) = \tau Q(x, y, z)$ for rational τ, then $a = \tau b, b = \tau c$ and $c = \tau a$. This means that $a = \tau^3 a$, $b = \tau^3 b$ and $c = \tau^3 c$, and since a, b, c cannot all be zero, we must have $\tau^3 = 1$, so $\tau = 1$.

Thus $P(x, y, z)$ has a factor $Q(x, y, z)Q(y, z, x)Q(z, x, y)$ of degree 3 that factors into three factors of degree 1, which contradicts the earlier statement that it has an irreducible factor of degree at least $n - 2$. We conclude that a, b, c must all be equal. So the degree one factor is $x + y + z$, which, as in the previous solution, is only a factor of $P(x, y, z)$ for $n = 3$.

Case 2: $P(x, y, z)$ has an irreducible factor of degree 2 *(and no irreducible factor of degree 1).*

There is an irreducible degree 2 factor $Q(x, y, z)$ such that $Q(x, y, y) = (x - y)^2$ and there are no irreducible factors of degree 1. Using a similar symmetry argument to the previous case, we can see that

$$Q(x, y, z) = \lambda(x^2 + y^2 + z^2) + \mu(xy + yz + zx)$$

for integers λ and μ. Since $Q(x, y, y) = (x - y)^2$, we have that $\lambda = 1$ and $\mu = -1$. Thus,

$$Q(x, y, z) = x^2 + y^2 + z^2 - (xy + yz + zx) = (z + \omega x + \omega^2 y)(z + \omega^2 x + \omega y),$$

where ω is a primitive cubic root of unity. Thus,

$$P(x, y, -\omega x - \omega^2 y) = Q(x, y, -\omega x - \omega^2 y)R(x, y, -\omega x - \omega^2 y) = 0.$$

In particular, $P(x, 0, -\omega x) = 0$, so $x^n + (-\omega x)^n = 0$ for all x, which can only happen if $n \equiv 3 \pmod 6$.

In the case when $n \equiv 3 \pmod 6$ and $n \geqslant 9$ (the latter inequality is because we are assuming $n \geqslant 4$, having dealt with the cases of $n = 2, 3$ separately), if we consider $T(x, y) = P(x, y, -\omega x - \omega^2 y)$, we get

$$T(x, y) = x^n + y^n - (x + \omega y)^n + \omega x^{n-2} y(x + \omega y) + \omega x y^{n-2}(x + \omega y) + \omega x y(x + \omega y)^{n-2}.$$

For $n \geqslant 4$, the coefficient of $x^{n-1}y$ is $-(n - 2)\omega \neq 0$, so $T(x, y) \neq 0$, which is a contradiction. Thus, this case is impossible.

Comment. Over $\mathbb{C}$, the polynomial $P(x, y, z)$ for $n = 2$ factors as

$$P(x, y, z) = (x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z),$$

where ω is again a primitive cubic root of unity. For $n \geqslant 4$, it turns out that the polynomial $P(x, y, z)$ is also irreducible over $\mathbb{C}$. This can be seen as follows. By Gauss' Lemma applied to $\mathbb{Z}[y, z]$, the polynomial $P(x, y, z)$ being irreducible over $\mathbb{Z}$ implies it being irreducible over $\mathbb{Q}$. Then its indecomposable factors in the unique factorisation domain $\mathbb{C}[x, y, z]$ must be Galois conjugate to each other, and hence it must be of the form

$$P(x, y, z) = \prod_{\sigma\colon K \hookrightarrow \mathbb{C}} {}^{\sigma}Q(x, y, z),$$

where $Q(x, y, z)$ is a monic polynomial in x, irreducible over $\mathbb{C}$, and K is the number field generated by the coefficients of Q. We now formally expand P around the point $(1, 1, 1)$ and compute that

$$P(1 + x, 1 + y, 1) = (2n - 3)(x^2 - xy + y^2) + (\text{higher order terms}) = \prod_{\sigma\colon K \hookrightarrow \mathbb{C}} {}^{\sigma}Q(1 + x, 1 + y, 1).$$

Because ${}^{\sigma}Q(1, 1, 1) = 0$ for at least one σ, it is zero for all σ. Assuming for the sake of contradiction that $K \neq \mathbb{Q}$, we conclude that $[K : \mathbb{Q}] = 2$. Moreover, the lowest degree terms of $Q(1 + x, 1 + y, 1)$ must be of the form $a(x + \omega^m y)$ for $1 \leqslant m \leqslant 2$, and hence $K = \mathbb{Q}(\omega)$. At this point, we set $x = 0$ and $y = z = 1$ to obtain

$$P(0, 1, 1) = 2 = Q(0, 1, 1)^{\sigma}Q(0, 1, 1),$$

where σ now denotes the nontrivial automorphism of $\mathbb{Q}(\omega)$. But we check that there is no $\alpha \in \mathbb{Q}(\omega)$ satisfying $\alpha\sigma(\alpha) = 2$, showing that P is irreducible.

A6. Let S be a set of positive integers, possibly infinite, such that no positive integer greater than 1 divides all elements of S. Determine all non-periodic infinite sequences $a_1, a_2, a_3, \ldots$ of positive integers such that, for all positive integers n,

- $a_n \leqslant |a_{n+\ell} - \ell|$ for all ℓ in S, and
- $a_n = |a_{n+\ell} - \ell|$ for at least one ℓ in S.

We say that an infinite sequence $a_1, a_2, a_3, \ldots$ is periodic *if there exists a positive integer t such that $a_n = a_{n+t}$ for all positive integers n.*

(Singapore)

Answer: The only sequences satisfying the conditions of the problem are of the form $a_n = n + c$ for an integer $c \geqslant 0$.

Solution 1. Sequences of the type $a_n = n + c$ work because

$$a_n = n + c = |n + \ell + c - \ell| = |a_{n+\ell} - \ell| \text{ for all } \ell \text{ in } S.$$

We now show that these are the only sequences satisfying the desired properties. Let S be a set of integers and $a_1, a_2, a_3, \ldots$ a non-periodic sequence of integers satisfying the conditions in the problem statement. The proof is split into different claims.

Claim 1.1. The sequence $a_1, a_2, a_3, \ldots$ is unbounded.

Proof. Suppose for the sake of contradiction that it is bounded, that is, there exists a constant C such that $a_n \leqslant C$ for all $n \geqslant 1$. Then, for all $\ell > 2C$ in S,

$$a_n < \ell - a_{n+\ell} = |a_{n+\ell} - \ell|.$$

Therefore, for every $n \geqslant 1$, the elements ℓ in S for which $a_n = |a_{n+\ell} - \ell|$ (there is at least one) satisfy $\ell \leqslant 2C$.

For each $n \geqslant 1$, define the tuple $L_n = (a_{n+1}, a_{n+2}, \ldots, a_{n+2C})$. Using the equality condition repeatedly, the tuple L_n uniquely characterises the value of a_n, and hence L_{n-1}. Therefore, inductively, L_n determines the values of L_m for all $m \leqslant n$.

Let L be a tuple such that $L = L_n$ for infinitely many n. This exists by the infinite pigeonhole principle, as the number of possible tuples is at most C^{2C}. Let $n_1 \leqslant n_2 \leqslant \ldots$ be the subsequence of indices with $L_{n_i} = L$. Because of the previous observation, $n_{i+1} - n_i$ is constant and so $a_1, a_2, a_3, \ldots$ is periodic with period $n_2 - n_1$. This is a contradiction, hence the sequence cannot be bounded. □

Let T be a finite subset of S whose elements have greatest common divisor 1. Denote by M the largest element of T. Bézout's Lemma gives the following:

Fact. There exists a positive integer K such that all integers larger than K can be expressed as the sum of (not necessarily distinct) elements of T.*

Claim 1.2. There exists a positive integer N such that $a_n > M$ for all $n > N$.

Proof. Note that if $a_n \leqslant M$, then for all $t \in T$, we have that $a_{n-t} \leqslant |a_n - t| \leqslant M$ since $t \leqslant M$. Thus, $a_{n-t_1-\ldots-t_j} \leqslant M$ for all $t_1, \ldots, t_j \in T$. This means that $a_{n-k} \leqslant M$ for all $k > K$ and so, by definition of K, $a_m \leqslant M$ for every $m < n - K$. Hence, if $a_n \leqslant M$ for infinitely many n, then $a_n \leqslant M$ for all n, contradicting Claim 1.1. □

*To prove this, let $T = \{t_1, t_2, \ldots, t_k\}$ and $K = (t_1 + t_2 + \ldots + t_k)^2$. Suppose $m > K$. The generalised Bézout's Lemma implies that there exist $w_1, w_2, \ldots, w_k \in \mathbb{Z}$ such that $m = w_1 t_1 + w_2 t_2 + \cdots + w_k t_k$. Suppose $w_j < 0$. Then the inequlity $m > K$ implies that there exist j with $w_j > t_i$. Replace (w_i, w_j) by $(w_i + t_j, w_j - t_i)$. This gives another solution where the sum of the negative w_i has increased. Repeating this process must terminate at a solution to $m = w_1 t_1 + w_2 t_2 + \cdots + w_k t_k$ with all $w_i \geqslant 0$.

We introduce the sequence $b_n = a_n - n$.

Claim 1.3. We have that $b_n \leqslant b_{n+k}$ for all $n > N$ and $k > K$.

Proof. If $n > N$ and $t \in T$, then it cannot be the case that $a_{n+t} \leqslant t - a_n$, as this would mean that $a_{n+t} < t - M \leqslant 0$, a contradiction. So, the condition $a_n \leqslant |a_{n+t} - t|$ becomes $a_n \leqslant a_{n+t} - t$ when $n > N$. The latter can be rewritten as $b_n \leqslant b_{n+t}$ for all $n > N$ and $t \in T$. Thus, by definiton of K, we have that $b_n \leqslant b_{n+k}$ for all $k > K$. □

Claim 1.4. The sequence b_n is eventually constant.

Proof. We will repeatedly use the following observation: if there exists an integer B such that $b_m \leqslant B$ for infinitely many indices m, then $b_1, b_2, b_3, \ldots$ is eventually bounded by B. Indeed, for all $n > N$, there exists $m > n + K$ with $b_m \leqslant B$, which means that $m - n > K$ and therefore $b_n \leqslant b_{n+(m-n)} = b_m \leqslant B$ by Claim 1.3.

Next, we prove that $b_1, b_2, b_3, \ldots$ is bounded. If the sequence was not bounded, then it would also not be eventually bounded by any integer B. Taking B to be positive and larger than some elements in the sequence $b_1, b_2, b_3, \ldots$, the contrapositive of the previous observation would imply that there is an index m such that $b_m \leqslant B$ and $b_n > B$ for all $n > m$. Now, there exists ℓ in S such that $a_m = |a_{m+\ell} - \ell| = |b_{m+\ell} + m| = b_{m+\ell} + m$. Hence, $b_{m+\ell} = b_m \leqslant B$ which is a contradiction.

Because $b_1, b_2, b_3, \ldots$ is bounded, if it was not eventually constant, then the sequence would take at least two different values infinitely many times. But this contradicts the initial observation. □

To conclude, Claim 1.4 implies that there exists a positive integer L and an integer c such that $a_n = n + b_n = n + c$ for all $n \geqslant L$. Let L be the smallest positive integer satisfying this property. If $L \geqslant 2$, then there would exist ℓ in S such that $a_{L-1} = |a_{L-1+\ell} - \ell| = |c + L - 1|$. Since $c = b_L = a_L - L$, then $c + L \geqslant 1$ and thus $a_{L-1} = L - 1 + c$, a contradiction to the minimality of L. So $L = 1$ and $a_n = n + c$ for every $n \geqslant 1$. Since $a_1 \geqslant 1$, we must have $c \geqslant 0$ which concludes the proof.

Solution 2. As in Solution 1, we start by proving that the sequence $a_1, a_2, a_3, \ldots$ is unbounded and use it to prove the following stronger statement:

Claim 2.1. For any positive integers n and m, the subsequence $a_n, a_{n+m}, a_{n+2m}, \ldots$ is unbounded.

Proof. Suppose otherwise, that there exist such bounded subsequences. Let m be the smallest positive integer for which there exists n such that $a_n, a_{n+m}, a_{n+2m}, \ldots$ is bounded. Note that $m > 1$, else the sequence $a_1, a_2, a_3, \ldots$ will be bounded.

Take some ℓ in S such that $d = \gcd(\ell, m) < m$. This must exist since the gcd of all the elements of S is 1. Since m is assumed to be minimal, the subsequence $a_n, a_{n+d}, a_{n+2d}, \ldots$ will be unbounded. Pick $N \equiv n \pmod{d}$ such that $a_N > \ell$. Note that $a_N \leqslant |a_{N+\ell} - \ell|$. Since $a_N > \ell$, this means that $a_{N+\ell} > \ell$ as well, which means that $a_{N+\ell} \geqslant a_N + \ell$. A simple induction shows that $a_{N+k\ell} \geqslant a_N + k\ell$ for all positive integers k.

By Bézout's Lemma and since $d = \gcd(\ell, m) \mid N - n$, there exist infinitely many positive integers i, j such that $jm - i\ell = N - n$ and thus $a_{n+jm} = a_{N+i\ell} \geqslant a_N + i\ell$. This means that a_{n+jm} is unbounded, which gives a contradiction. □

Claim 2.2. For all ℓ in S, there exists $c_\ell \geqslant 1$, such that for every $n \geqslant c_l$, we have $a_{n+\ell} \geqslant a_n + \ell$.

Proof. Take any residue $1 \leqslant r \leqslant \ell$ modulo ℓ. By Claim 2.1, the subsequence $a_r, a_{r+\ell}, a_{r+2\ell}, \ldots$ is unbounded. Hence, there exists $a_{r+k\ell} > \ell$. Since $a_{r+k\ell} \leqslant |a_{r+(k+1)\ell} - \ell|$, this forces $a_{r+(k+1)\ell} > \ell$ as well, meaning that $a_{r+(k+1)\ell} \geqslant a_{r+k\ell} + \ell$. A simple induction shows that for any $m \geqslant k$, $a_{r+(m+1)\ell} \geqslant a_{r+m\ell} + \ell$.

Hence, $a_{n+\ell} \geqslant a_n + \ell$ for sufficiently large n in each residue class modulo ℓ, meaning that $a_{n+\ell} \geqslant a_n + \ell$ for all sufficiently large n. □

By the property of S, there exist numbers $\ell_1, \ell_2, \ldots, \ell_k$ in S such that $\gcd(\ell_1, \ell_2, \ldots, \ell_k) = 1$. As in the Fact in Solution 1, we note that any sufficiently large m can be represented as $m = w_1\ell_1 + w_2\ell_2 + \cdots + w_k\ell_k$ for some nonnegative integers $w_1, w_2, \ldots, w_k$.

Claim 2.3. There exists some positive integer N such that, for all $m, n \geqslant N$, we have $a_{n+m} \geqslant a_n + m$. In particular, we have $a_n \geqslant n - N$ for every $n \geqslant 2N$.

Proof. We take N larger than $c_{\ell_1}, c_{\ell_2}, \ldots, c_{\ell_k}$ from Claim 2.2 and large enough such that every $m \geqslant N$ can be written as $w_1\ell_1 + w_2\ell_2 + \cdots + w_k\ell_k$ for some nonnegative integers $w_1, w_2, \ldots, w_k$. For $n, m \geqslant N$, writing $m = w_1\ell_1 + w_2\ell_2 + \cdots + w_k\ell_k$ and applying Claim 2.2 several times, we have

$$a_{n+m} = a_{n+\sum_{i=1}^{k} w_i\ell_i} \geqslant a_n + \sum_{i=1}^{k} w_i\ell_i = a_n + m. \qquad \square$$

Claim 2.4. For every $n \geqslant 2N$ and every $m \geqslant N$, we have $a_{n+m} = a_n + m$.

Proof. Fix $n \geqslant 2N$. We first show that there are infinitely many m such that $a_{n+m} = a_n + m$. First, there exists s_1 in S such that $a_n = |a_{n+s_1} - s_1|$. Since $n + s_1 \geqslant 2N$, it follows by Claim 2.3 that $a_{n+s_1} \geqslant n + s_1 - N > s_1$, so that $a_n = a_{n+s_1} - s_1$. Similarly, there exists s_2 in S for which $a_{n+s_1+s_2} = a_{n+s_1} + s_2 = a_n + s_1 + s_2$. Continuing this process, we find infinitely many m such that $a_{n+m} = a_n + m$.

Recall that if $m \geqslant N$, then $a_{n+m} \geqslant a_n + m$ by Claim 2.3. Given $m \geqslant N$, there exists $M \geqslant N + m$ such that $a_{n+M} = a_n + M$. Then,

$$a_{n+M} \geqslant a_{n+m} + M - m \geqslant a_n + m + M - m = a_n + M.$$

However, since $a_{n+M} = a_n + M$, the inequalities must be equalities. Hence, $a_{n+m} = a_n + m$ for every $n \geqslant 2N$ and every $m \geqslant N$. $\square$

Claim 2.5. For any $n \geqslant 2N$, $a_{n+1} = a_n + 1$.

Proof. If $m \geqslant N$, then by Claim 2.4, $a_n + m + 1 = a_{n+m+1} = a_{n+1} + m$ which gives the desired result. $\square$

We now conclude in the same way as in Solution 1. Claim 2.5 gives $a_{n+1} = a_n + 1$ for all $n \geqslant 2N$. In particular, $a_{2N-1+\ell} = a_{2N} - 1 + \ell \geqslant \ell$ for every ℓ in S. Hence,

$$a_{2N-1} = \min_{\ell \in S}(a_{2N-1+\ell} - \ell) = \min_{\ell \in S}(a_{2N} - 1 + \ell - \ell) = a_{2N} - 1.$$

Repeating this argument, we have $a_{n+1} = a_n + 1$ for all positive integers n. Therefore, $a_n = n + c$ for some integer constant c. Since $a_1 \geqslant 1$, we must have $c \geqslant 0$ which completes the solution.

A7. For each integer $k \geqslant 3$, prove that there exists a unique tuple $(x_1, x_2, \ldots, x_k)$ of positive real numbers such that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k$ and

$$\left\lfloor nx_1 + \frac{1}{2} \right\rfloor + \left\lfloor nx_2 + \frac{1}{2} \right\rfloor + \cdots + \left\lfloor nx_k + \frac{1}{2} \right\rfloor = n$$

for every integer n.

Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. For example, $\lfloor -\pi \rfloor = -4$ and $\lfloor 2 \rfloor = \lfloor 2.9 \rfloor = 2$.

(Thailand)

Solution 1. We start with some observations. First of all, note that, for every real number x,

$$\left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor - \lfloor x \rfloor. \tag{1}$$

Also observe that if a tuple $(x_1, x_2, \ldots, x_k)$ satisfies the conditions of the problem statement, and since $nx - 1/2 < \lfloor nx + 1/2 \rfloor \leqslant nx + 1/2$, then dividing both sides of the equation by n and letting n go to infinity gives

$$x_1 + x_2 + \cdots + x_k = 1.$$

In particular, all numbers x_i belong to $(0, 1)$. Inspired by equation (1), we will consider the binary expansion of the fractional part of a real number x which we write

$$\{x\} = x - \lfloor x \rfloor = \frac{\langle x \rangle_1}{2} + \frac{\langle x \rangle_2}{2^2} + \cdots,$$

where $\langle x \rangle_i \in \{0, 1\}$ is the i^{th} binary digit. Note that

$$\langle x \rangle_1 = \left\lfloor x + \frac{1}{2} \right\rfloor - \lfloor x \rfloor = \lfloor 2x \rfloor - 2\lfloor x \rfloor. \tag{2}$$

Existence. We first show the existence of a tuple $(x_1, x_2, \ldots, x_k)$ with the given property. The desired tuple turns out to have the property that, for every $i \geqslant 1$, there exists a unique $j \in \{1, \ldots, k\}$ such that $\langle x_j \rangle_i = 1$. This immediately implies $x_1 + \cdots + x_k = 1$. The distribution of binary digits we want to consider is the following.

$\langle x_j \rangle_i$	1	2	$\cdots$	k	$k+1$	$k+2$	$\cdots$	$2k$	$\cdots$
x_1	1	0	$\cdots$	0	1	0	$\cdots$	0	1
x_2	0	1		$\vdots$	0	1		$\vdots$	0
$\vdots$	$\vdots$		$\ddots$	0	$\vdots$		$\ddots$	0	$\vdots$
x_k	0	$\cdots$	0	1	0	$\cdots$	0	1	0

In other words, we claim that the tuple

$$(x_1, x_2, \ldots, x_{k-1}, x_k) = \left(\frac{2^{k-1}}{2^k - 1}, \frac{2^{k-2}}{2^k - 1}, \ldots, \frac{2}{2^k - 1}, \frac{1}{2^k - 1} \right) \tag{3}$$

has the given property. To verify the claim, we use equation (1) and $x_i = 2x_{i+1}$ to compute

$$\sum_{i=1}^{k} \left\lfloor nx_i + \frac{1}{2} \right\rfloor = \sum_{i=1}^{k} \left(\lfloor 2nx_i \rfloor - \lfloor nx_i \rfloor \right) = \left\lfloor \frac{n2^k}{2^k - 1} \right\rfloor - \left\lfloor \frac{n}{2^k - 1} \right\rfloor = \left\lfloor n + \frac{n}{2^k - 1} \right\rfloor - \left\lfloor \frac{n}{2^k - 1} \right\rfloor = n.$$

Uniqueness. Let $(x_1, x_2, \ldots, x_k)$ be a tuple that satisfies the conclusion of the problem statement. In particular $x_1 + x_2 + \cdots + x_k = 1$ and all the numbers x_i belong to $(0, 1)$. The first milestone is to prove that for every $i \geqslant 1$, there exists a unique $j \in \{1, 2, \ldots, k\}$ such that $\langle x_i \rangle_j = 1$.

We start with some notation. For positive integers n and r, define

$$c(n, r) = \langle nx_1 \rangle_r + \langle nx_2 \rangle_r + \cdots + \langle nx_k \rangle_r.$$

Note that $c(n, r) = c(n2^{r-1}, 1)$ since $\langle x \rangle_r = \langle 2^{r-1}x \rangle_1$. We now study the properties of $c(n, r)$.

Claim 1.1. The value of $c(n, r)$ does not depend on r.

Proof. Combining (1) and (2), we obtain

$$\left\lfloor 2x + \frac{1}{2} \right\rfloor - 2 \left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor + \langle 2x \rangle_1 - 2(\lfloor x \rfloor + \langle x \rangle_1) = \langle 2x \rangle_1 - \langle x \rangle_1. \tag{4}$$

Summing (4) evaluated at $x = nx_1, nx_2, \ldots, nx_k$ gives $c(2n, 1) = c(n, 1)$. Combining this with $c(n, r) = c(n2^{r-1}, 1)$ gives the result. □

We denote by c_n the common value of $c(n, r)$ across all values of r. Recall from the proof of Claim 1.1 that $c_{2n} = c_n$.

Claim 1.2. We have $c_1 = 1$ and $c_{m+n} \leqslant c_m + c_n$ for all positive integers m, n. Moreover, if $c_{m+n} = c_m + c_n$, then $\langle mx_i \rangle_r + \langle nx_i \rangle_r \leqslant 1$ for every i and r.

Proof. Summing (2) evaluated at $nx_1, nx_2, \ldots, nx_k$ gives

$$c_n = \sum_{i=1}^{k} \left(\left\lfloor nx_i + \frac{1}{2} \right\rfloor - \lfloor nx_i \rfloor \right) = n - \sum_{i=1}^{k} \lfloor nx_i \rfloor. \tag{5}$$

As $x_1, x_2, \ldots, x_k < 1$, we have $c_1 = 1$. Combining (5) with the inequality $\lfloor mx_i \rfloor + \lfloor nx_i \rfloor \leqslant \lfloor (m + n)x_i \rfloor$, we have $c_{m+n} \leqslant c_m + c_n$. If $c_{m+n} = c_m + c_n$, then $c_{m2^r+n2^r} = c_{m2^r} + c_{n2^r}$ for every $r \geqslant 0$. Therefore, we must have the equality

$$\lfloor m2^r x_i \rfloor + \lfloor n2^r x_i \rfloor = \lfloor (m + n)2^r x_i \rfloor \tag{6}$$

for every i and $r \geqslant 0$. Equation (6) implies that $\langle mx_i \rangle_r + \langle nx_i \rangle_r \leqslant 1$, as desired. □

Note that $c_1 = 1$ can be reformulated by saying that, for every $r \geqslant 1$,

$$\langle x_1 \rangle_r + \langle x_2 \rangle_r + \cdots + \langle x_k \rangle_r = 1.$$

In other words, for every $r \geqslant 1$, there is a unique index $a_r \in \{1, 2, \ldots, k\}$ such that $\langle x_{a_r} \rangle_r = 1$. This is the first milestone. Next, in order to prove that the tuple $(x_1, x_2, \ldots, x_k)$ is the tuple (3), we shall prove that a_r is the unique element of $\{1, 2, \ldots, k\}$ congruent to r modulo k.

Whenever consecutive terms of the sequence $a_1, a_2, a_3, \ldots$ coincide, we group them into *segments*. Denote a segment by a triplet (ν, t, ℓ) where ν is the repeating value in the sequence, t is the first index of the repetition and ℓ is the length of the segment. We will call ν the *value* of the segment. For example, the first three segments in the sequence $3, 1, 2, 2, 2, 2, 3, \ldots$ are denoted by $(3, 1, 1)$, $(1, 2, 1)$ and $(2, 3, 4)$. We now make a few claims about segments.

Claim 1.3. Apart from the first segment, all segments of value ν are preceded by segments of the same value.

Proof. Suppose there are two segments (ν, t_1, ℓ_1) and (ν, t_2, ℓ_2) with $1 < t_1 < t_2$ that are preceded by segments of value ν_1 and ν_2 with $\nu_1 \neq \nu_2$. In particular, both ν_1 and ν_2 are different from ν. We study the $(t_1-1)^{\text{st}}$, $(t_1)^{\text{th}}$, $(t_2-1)^{\text{st}}$ and $(t_2)^{\text{th}}$ binary digits of $x_\nu, x_{\nu_1}, x_{\nu_2}$. Depending on these values, we get lower and upper bounds on $\{2^{t_1-2}x_i\}$ and $\{2^{t_2-1}x_i\}$ for $i \in \{\nu, \nu_1, \nu_2\}$.

x	$\langle x\rangle_{t_1-1}$	$\langle x\rangle_{t_1}$	lower bound on $\{2^{t_1-2}x\}$	upper bound on $\{2^{t_1-2}x\}$	$\langle x\rangle_{t_2-1}$	$\langle x\rangle_{t_2}$	lower bound on $\{2^{t_2-2}x\}$	upper bound on $\{2^{t_2-2}x\}$
x_ν	0	1	1/4	1/2	0	1	1/4	1/2
x_{ν_1}	1	0	1/2	3/4	0	0	0	1/4
x_{ν_2}	0	0	0	1/4	1	0	1/2	3/4

Note that the upper bounds are strict. Letting $n = 2^{t_1-2} + 2^{t_2-2}$, it follows that $\frac{1}{2} \leqslant \{nx_i\} < 1$ for $i \in \{\nu, \nu_1, \nu_2\}$. Hence, $\langle nx_i\rangle_1 = 1$ for $i \in \{\nu, \nu_1, \nu_2\}$. This implies that $c_n \geqslant 3$. However, by Claim 1.2, $c_n \leqslant c_{2^{t_1-2}} + c_{2^{t_2-2}} = c_1 + c_1 = 2$, a contradiction. □

Since $0 < x_i < 1$ for all i, all values $\nu \in \{1, 2, \ldots, k\}$ must appear as the value of some segment. By Claim 1.3, every value determines the preceding value uniquely. This means that the values of the k first segments are a permutation of $\{1, 2, \ldots, k\}$ and that they repeat periodically after that.

Claim 1.4. All segments, except possibly for the first segment, have equal length ℓ, and the length of the first segment is at most ℓ.

Proof. Let ℓ be the minimal length of a non-initial segment, and let (ν, t, ℓ) be a non-initial segment of length ℓ. Let (ν_0, t_0, ℓ_0) be the segment preceding (ν, t, ℓ). Because each segment has length at least ℓ, and $k \geqslant 3$, segments of the same value must be separated by at least 2ℓ entries. We study the binary digits of x_ν and x_{ν_0}, first starting at the $(t-1)^{\text{st}}$ digit, and then at the $(t+\ell-1)^{\text{st}}$ digit.

x	$\langle x\rangle_{t-1}, \langle x\rangle_t, \ldots$	lower bound on $\{2^{t-2}x\}$	upper bound on $\{2^{t-2}x\}$
x_ν	$0\,\underbrace{1\cdots1}_{\ell \text{ ones}}\,\underbrace{0\cdots0}_{2\ell \text{ zeros}}$	1/4	$1/2 - 2^{-\ell-1} + 2^{-3\ell-1}$
x_{ν_0}	$10\cdots$	1/2	3/4

x	$\langle x\rangle_{t+\ell-1}, \langle x\rangle_{t+\ell}, \ldots$	lower bound on $\{2^{t+\ell-2}x\}$	upper bound on $\{2^{t+\ell-2}x\}$
x_ν	$1\,\underbrace{0\cdots0}_{2\ell \text{ zeros}}$	1/2	$1/2 + 2^{-2\ell-1}$
x_{ν_0}	$\underbrace{0\cdots0}_{\ell+1 \text{ zeros}}$	0	$2^{-\ell-1}$

We let $n = 2^{t-2} + 2^{t+\ell-2}$. Since $\ell \geqslant 1$, we have $(1/2 - 2^{-\ell-1} + 2^{-3\ell-1}) + (1/2 + 2^{-2\ell-1}) < 1$. Therefore, $\frac{1}{2} \leqslant \{nx_\nu\}, \{nx_{\nu_0}\} < 1$ and $\langle nx_\nu\rangle_1 = \langle nx_{\nu_0}\rangle_1 = 1$. As a result, we have the tight inequality

$$2 \leqslant c_n = c_{2^\ell+1} \leqslant c_{2^\ell} + c_1 = 2.$$

So, if there were a segment (ν', t', ℓ') with $\ell' > \ell$, then $\langle x_{\nu'}\rangle_{t'} = \langle 2^\ell x_{\nu'}\rangle_{t'} = 1$, contradicting the equality condition from Claim 1.2 for $c_{2^\ell+1} = c_{2^\ell} + c_1$. □

To finish the proof, it remains to show that $\ell = 1$. Let λ be the length of the first segment. Claims 1.3 and 1.4 imply that some permutation of $\{2^\lambda x_1\}, \{2^\lambda x_2\}, \ldots, \{2^\lambda x_{k-1}\}, \{2^\lambda x_k\}$ is equal to

$$\left(\frac{2^\ell-1}{2^{k\ell}-1}\cdot 2^{(k-1)\ell}, \frac{2^\ell-1}{2^{k\ell}-1}\cdot 2^{(k-2)\ell}, \ldots, \frac{2^\ell-1}{2^{k\ell}-1}\cdot 2^\ell, \frac{2^\ell-1}{2^{k\ell}-1}\cdot 1\right).$$

Since the binary expansion of $\frac{(2^{\ell+1}-1)(2^{\ell}-1)}{2^{k\ell}-1}$ is periodic with a period of $k\ell$ binary digits and the sum of binary digits of the repeating segment $(2^{\ell+1}-1)(2^{\ell}-1)$ is $\ell+1$, we have

$$\sum_{r=1}^{k\ell} \left\langle \frac{(2^{\ell+1}-1)(2^{\ell}-1)}{2^{k\ell}-1} \cdot 2^s \right\rangle_r = \ell+1$$

for all integers $s \geqslant 0$. Therefore, letting $n = 2^{\lambda}(2^{\ell+1}-1)$,

$$c_n = \frac{1}{k\ell}\sum_{r=1}^{k\ell} c(n,r) = \sum_{i=1}^{k}\sum_{r=1}^{k\ell} \left\langle \frac{(2^{\ell+1}-1)(2^{\ell}-1)}{2^{k\ell}-1} \cdot 2^{(i-1)\ell} \right\rangle_r = \frac{k(\ell+1)}{k\ell},$$

which is only an integer when $\ell = 1$. Using Claim 1.4, we conclude that $\lambda = 1$ and

$$(x_1, x_2, \ldots, x_{k-1}, x_k) = \left(\frac{2^{k-1}}{2^k-1}, \frac{2^{k-2}}{2^k-1}, \ldots, \frac{2}{2^k-1}, \frac{1}{2^k-1}\right).$$

Solution 2. Denote by $\mathbb{Z}+\frac{1}{2} = \{\ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots\}$. Note that for all real numbers x, we have that

$$\left\lfloor x+\frac{1}{2}\right\rfloor + \left\lfloor -x+\frac{1}{2}\right\rfloor = \begin{cases} 1 & \text{if } x \in \mathbb{Z}+\frac{1}{2}, \\ 0 & \text{if } x \notin \mathbb{Z}+\frac{1}{2}. \end{cases}$$

Summing this equality for $x = nx_1, nx_2, \ldots, nx_k$, we deduce that $nx_i \notin \mathbb{Z}+\frac{1}{2}$ for all n and $i \in \{1,2,\ldots,k\}$. This means that $\frac{a}{x_i}$ is never an integer for all $a \in \mathbb{Z}+\frac{1}{2}$ and $i \in \{1,2,\ldots,k\}$.

We consider the following process: for every integer N, we place a bucket at the interval $(N, N+1)$. Then, for each $a \in \mathbb{Z}+\frac{1}{2}$ and $i \in \{1,2,\ldots,k\}$, we drop a ball of colour i on the real line at $\frac{a}{x_i}$, which falls into the bucket corresponding to the interval that $\frac{a}{x_i}$ lies in. By construction, the configurations of balls in buckets is symmetric about 0.

Lemma 1. Assume that $mx_i \notin \mathbb{Z}+\frac{1}{2}$ for all $m \in \mathbb{Z}$ and $1 \leqslant i \leqslant k$. Then every bucket contains exactly one ball if and only if $\sum_{i=1}^{k}\lfloor nx_i+\frac{1}{2}\rfloor = n$ holds for all $n \in \mathbb{Z}$.

Proof. By symmetry between positive and negative buckets, every bucket contains exactly one ball if and only if exactly n balls land in $(0,n)$ for every $n \geqslant 1$. Similarly, by the argument above, the equation $\sum_i\lfloor nx_i+\frac{1}{2}\rfloor = n$ holds for all $n \in \mathbb{Z}$ if and only if it holds for all $n \geqslant 1$.

A ball of colour i lands in $(0,n)$ if and only if it was dropped at $\frac{a}{x_i}$ for $a \in \mathbb{Z}+\frac{1}{2}$ with $0 < \frac{a}{x_i} < n$. Thus, the number of balls of colour i dropped in $(0,n)$ is given by the largest integer less than $nx_i+\frac{1}{2}$, which is equal to $\lfloor nx_i+\frac{1}{2}\rfloor$ as $nx_i \notin \mathbb{Z}+\frac{1}{2}$. Summing over all colours then shows the desired result. □

Since each bucket contains exactly one ball, we may consider the resulting sequence of colours $\ldots, C_{-\frac{3}{2}}, C_{-\frac{1}{2}}, C_{\frac{1}{2}}, C_{\frac{3}{2}}, \ldots$, which we refer to as the *full sequence*, as well as the *positive sequence* $C_{\frac{1}{2}}, C_{\frac{3}{2}}, \ldots$, where $C_{N+\frac{1}{2}}$ is the colour of the ball in the bucket $(N, N+1)$. By symmetry between positive and negative buckets, we have that $C_a = C_{-a}$ for all $a \in \mathbb{Z}+\frac{1}{2}$. We first make some basic observations about the sequence of colours. In the sequence, a *block* of colour i is a maximal consecutive subsequence consisting only of i's, and a *gap* of colour i is a maximal consecutive subsequence containing no i's.

Lemma 2. For every $y, z \in \mathbb{Z}+\frac{1}{2}$, every positive integer ℓ and every colour i, the number of occurrences of i in the sequences $C_{y+1}, \ldots, C_{y+\ell}$ and $C_{z+1}, \ldots, C_{z+\ell}$ differ by at most 1. Consequently, any two blocks of i differ in length by at most 1, and any two gaps of i differ in length by at most 1.

Proof. The number of occurrences of i in $C_{y+1}, \ldots, C_{y+\ell}$ is the number of balls dropped into the interval $(y+\frac{1}{2}, y+\ell+\frac{1}{2})$, which is equal to the number of multiples of $\frac{1}{x_i}$ in the interval $(y+\frac{1}{2}+\frac{1}{2x_i}, y+\ell+\frac{1}{2}+\frac{1}{2x_i})$. The same holds for the sequence $C_{z+1}, \ldots, C_{z+\ell}$ with the interval $(z+\frac{1}{2}+\frac{1}{2x_i}, z+\ell+\frac{1}{2}+\frac{1}{2x_i})$ instead. Since these two intervals have the same length, the number of multiples of $\frac{1}{x_i}$ they contain differ by at most 1. □

In the positive sequence, the first occurrences of each colour happen in the order $1, 2, \ldots, k$ because $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k$. In particular, we have that $C_{\frac{1}{2}} = 1$ so $C_{-\frac{1}{2}} = 1$ as well, which means that $1 \in \{C_a, C_{a+1}\}$ for every $a \in \mathbb{Z} + \frac{1}{2}$ by Lemma 2.

In what follows, given sequences S_1 and S_2, we will denote by $S_1 * S_2$ the concatenation of S_1 and S_2. For $j \in \{1, 2, \ldots, k\}$, define the sequence T_j recursively by $T_1 = 1$ and $T_j = T_{j-1} * j * T_{j-1}$. Note that T_j has length $2^j - 1$ and has 2^{j-1} occurrences of 1. We will prove by induction on j the following two statements for all $j \leqslant k$:

- $A(j)$: The positive sequence begins with T_j.
- $B(j)$: For all $\ell > j$, any occurrence of ℓ in the sequence must be preceded and followed by T_j.

We already proved that $A(1)$ and $B(1)$ hold.

Claim 2.1. The statement $A(2)$ holds.

Proof. Suppose that in the positive colour sequence, the first occurrence of 2 happens after r occurrences of 1. Then by symmetry, the middle $2r+2$ terms of the full sequence are $\cdots 21 \cdots 12 \cdots$, so 1 has a block of length $2r$ and 2 has a gap of length $2r$. Consider now the first occurrence of 3 in the positive sequence, which must be preceded and followed by 1. Since blocks of 1 have length at least $2r-1$ by Lemma 2, there exists a subsequence $1 \cdots 131 \cdots 1$ of length $4r-1$. As gaps of 2 have length at most $2r+1$, this means that $2r+1 \geqslant 4r-1$, so $r \leqslant 1$ and $C_{\frac{3}{2}} = 2$. This implies $C_{\frac{5}{2}} = 1$, which proves $A(2)$. □

Claim 2.2. For $2 \leqslant j < k$, the statements $B(j-1)$ and $A(j)$ imply $B(j)$. (Note that $B(k)$ is tautologically true.)

Proof. Since $A(j)$ holds, the middle of the full sequence is $T_j * T_j = T_{j-1} * j * T_{j-1} * T_{j-1} * j * T_{j-1}$, which contains a gap $T_{j-1} * T_{j-1}$ of j of length $2^j - 2$. By $B(j-1)$, any occurrence of $\ell > j$ must be contained in a subsequence $T_{j-1} * \ell * T_{j-1}$. This subsequence has length $2^j - 1$ and contains no j. As gaps of j have length at most $2^j - 1$ by Lemma 2, $T_{j-1} * \ell * T_{j-1}$ must be preceded and followed by j. Applying $B(j-1)$ to these two occurrences of j, the subsequence $j * T_{j-1} * \ell * T_{j-1} * j$ must be contained in the subsequence $T_{j-1} * j * T_{j-1} * \ell * T_{j-1} * j * T_{j-1} = T_j * \ell * T_j$, which proves $B(j)$. □

Claim 2.3. For $2 \leqslant j < k$, the statements $A(j)$ and $B(j)$ imply $A(j+1)$.

Proof. Because $j+1 \leqslant k$, the colour $j+1$ appears somewhere in the full sequence, and, by $B(j)$, it is preceded by and followed by T_j. That is, $T_j * (j+1) * T_j$ appears as a subsequence of the full sequence. In particular, for every $i \leqslant j$ there is a gap $T_{i-1} * (j+1) * T_{i-1}$ of i of length $2^i - 1$.

By $A(j)$, the positive sequence starts with T_j, and we consider the next term $C_{2^j - \frac{1}{2}}$. If $C_{2^j - \frac{1}{2}} = i$ for some $2 \leqslant i \leqslant j$, then the sequence $T_j * i$ ends with the subsequence $i * T_{i-1} * i$. This contains a gap T_{i-1} of i of length $2^{i-1} - 1$, contradicting of Lemma 2 as $|(2^{i-1}-1) - (2^i - 1)| > 1$. Since $j+1$ has not appeared yet, we must have either $C_{2^j - \frac{1}{2}} = 1$ or $C_{2^j - \frac{1}{2}} = j+1$. If it is 1, then the subsequence $C_{-\frac{1}{2}}, \ldots, C_{2^j - \frac{1}{2}}$ is $1 * T_j * 1$, which has length $2^j + 1$ and $2^{j-1} + 2$ occurrences of 1. On the other hand, we have shown above that the sequence $T_j * (j+1) * T_j$ appears somewhere in the full sequence. This contains the subsequence $2 * 1 * (j+1) * T_j$, and removing the last term, we obtain a subsequence of length $2^j + 1$ with 2^{j-1} occurrences of

1. This contradicts Lemma 2, and thus $C_{2^j-\frac{1}{2}} = j+1$. Then $B(j)$ implies that this term is followed by T_j, and therefore the positive sequence begins with $T_j * (j+1) * T_j = T_{j+1}$, which proves $A(j+1)$. □

To conclude, we prove that the positive sequence is $T_k * T_k * \cdots$, i.e. it is periodic with repeating subsequence T_k. Since the positive sequence begins with T_k by $A(k)$, the middle of the full sequence is $T_k * T_k$. This has a gap of k of length $2^k - 2$, so gaps of k have length in $\{2^k-3, 2^k-2, 2^k-1\}$. By the arguments in the proof of Claim 2.3, any occurrence of T_k must be followed by 1. This precludes gaps of k of length 2^k-3 as the middle terms of such a gap would be $\cdots 3121213 \cdots$, where the 3121 is the end of a T_{k-1} following the first k and this T_{k-1} has to be followed by a 1. It also precludes gaps of k of length 2^k-1, as this gap would be $T_{k-1} * 1 * T_{k-1}$ which contains a block of 1's of length 3 in the middle. Thus, all gaps of k have length 2^k-2, which implies that the positive sequence is $T_k * T_k * \cdots$, as claimed.

Finally, note that in T_k, each colour i appears 2^{k-i} times. This means that $M2^{k-i}$ balls of colour i are dropped into $(0, M(2^k-1))$ for all M. Since we drop balls of colour i at $\frac{a}{x_i}$ for $a \in \mathbb{Z} + \frac{1}{2}$, the number of such balls dropped into $(0, N)$ divided by N converges to x_i as $N \to \infty$. This means that $x_i = \frac{2^{k-i}}{2^k-1}$ for all i.

To see that this k-tuple indeed satisfies the conditions, by Lemma 1, it suffices to show that exactly one ball is dropped into each bucket. Note that the ball $\frac{a}{x_i}$ is dropped into bucket $(N-1, N)$ if and only if the ball $\frac{a+2^{k-i}}{x_i}$ is dropped into bucket $(N+(2^k-1)-1, N+(2^k-1))$. This means that it suffices to show that exactly one ball is dropped into each of the buckets $(0,1),(1,2),\ldots,(2^k-2,2^k-1)$. Since $(2^k-1)x_i = 2^{k-i}$, the balls of colour i dropped into $(0, 2^k-1)$ are of the form $\frac{a}{x_i}$ for $a = (\mathbb{Z}+\frac{1}{2}) \cap (0, 2^{k-i})$. For $a = m - \frac{1}{2}$ with $1 \leqslant m \leqslant 2^{k-i}$, such a ball is dropped at $\frac{a}{x_i} = \frac{(2^k-1)(2m-1)}{2^{k-i+1}} = 2^{i-1}(2m-1) - \frac{2m-1}{2^{k-i+1}} \in (2^{i-1}(2m-1)-1, 2^{i-1}(2m-1))$. Hence, the balls of colour i are dropped in the buckets $(N-1, N)$ with $N \leqslant 2^k-1$ and $v_2(N) = i-1$. Considering all colours i, each of the buckets $(0,1),(1,2),\ldots,(2^k-2,2^k-1)$ have exactly one ball, as desired.

Comment. For $k=1$, $x_1 = 1$ is the unique solution to the condition. For $k=2$, uniqueness fails, and the solutions are given by $(x_1, x_2) = (a, 1-a)$ for $a \in (0, \frac{1}{2})$ not of the form $\frac{p}{q}$ for an odd integer p and an even integer q. This essentially amounts to the condition that $mx_1, mx_2 \notin \mathbb{Z} + \frac{1}{2}$ for all $m \in \mathbb{Z}$. In this case, the fact that every bucket contains exactly one ball if they are dropped at $\frac{a}{x_i}$ for $a \in \mathbb{Z} + \frac{1}{2}$ and $i = 1, 2$ is related to Beatty's Theorem about complementary Beatty sequences. However, knowledge of Beatty's Theorem does not seem to help with solving the problem for $k \geqslant 3$.

A8. Tim and Tam play a game. To start the game, Tim writes some nonzero real numbers, not necessarily distinct, on a blackboard. In each round, the following happens:

- First, Tam chooses a polynomial $P_k(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ whose coefficients $a_k, a_{k-1}, \ldots, a_1, a_0$ are all of the numbers currently written on the blackboard in some order. (For example, if the numbers on the blackboard are $4, -3, 4$, then $k = 2$ and Tam may choose the polynomial $4x^2 - 3x + 4$ or $-3x^2 + 4x + 4$ but not $4x - 3$ or $-3x^2 - 3x + 4$.)
- Then, if the equation $P_k(x) = 0$ has no real solutions, the game is stopped. Otherwise, Tim chooses a real number r such that $P_k(r) = 0$ and writes it on the blackboard, so there is now one more number on the blackboard.

Determine whether Tim can ensure that the game is never stopped, no matter what Tam does.

(France)

Answer: No, Tim cannot ensure that the game is never stopped.

Common remarks. In each round, we will omit the subscript and call the polynomial that Tam writes $P(x)$. In brief, Tam's strategy proceeds as follows:

(1) By choosing the coefficient of x^{2j} in $P(x)$ to be larger than its neighbouring coefficients for each j, prevent $P(x)$ from having negative roots and ensure that all numbers written on the board after a certain point are positive.

(2) By choosing the coefficients in step (1) more carefully, ensure that the numbers written on the board are not only positive but bounded away from 0 infinitely often. This allows Tam to build up a bank of "big positive numbers".

(3) By choosing the coefficients in step (1) so that each negative coefficient is surrounded by many big positive numbers, prevent $P(x)$ from having positive roots as well, stopping the game.

Solution 1. We begin with some auxiliary results for the above strategy. Define the functions $\mathcal{F}$ and $\mathcal{G}$ mapping sequences $a_0, \ldots, a_{2n}$ of odd length to sets of real numbers as follows:

$$\mathcal{F}(a_0, \ldots, a_{2n}) = \left\{ t : \sum_{i=0}^{2n} a_i t^i > 0 \right\} \quad \text{and} \quad \mathcal{G}(a_0, \ldots, a_{2n}) = \left\{ t : \frac{a_0 + a_{2n} t^{2n}}{2} + \sum_{i=1}^{2n-1} a_i t^i \geqslant 0 \right\}.$$

Note that if $a_0, a_{2n} > 0$, then $\mathcal{G}(a_0, \ldots, a_{2n}) \subseteq \mathcal{F}(a_0, \ldots, a_{2n})$. If the numbers on the board at any point are $a_0, \ldots, a_{2n}$ in some order with $a_0, a_{2n} > 0$ and $\mathcal{G}(a_0, \ldots, a_{2n}) = \mathbb{R}$, then $\mathcal{F}(a_0, \ldots, a_{2n}) = \mathbb{R}$ as well and Tam can choose $P(x) = \sum_{i=0}^{2n} a_i x^i$ to stop the game.

For $A > 0$, say that a sequence $a_0, \ldots, a_{2n}$ is *A-separated* if $a_i \geqslant A$ for even i and $a_i \leqslant A$ for odd i. If a sequence $a_0, \ldots, a_{2n}$ is A-separated for some $A > 0$, say that it is *separated*. Note that any sequence of nonzero reals of odd length with more positive numbers than negative numbers can be reordered to be separated.

Lemma 1.1. If A and B are positive reals and $a_0, \ldots, a_{2n}$ is an A-separated sequence with $a_i \geqslant -B$ for all i, then $t \in \mathcal{G}(a_0, \ldots, a_{2n}) \subseteq \mathcal{F}(a_0, \ldots, a_{2n})$ for all $t \leqslant \frac{A}{2B}$.

Proof. Let $P(x) = \frac{a_0+a_{2n}x^{2n}}{2} + \sum_{i=1}^{2n-1} a_i x^i$. For $t < 0$, we have that

$$P(t) \geqslant \frac{A + At^{2n}}{2} + \sum_{i=1}^{2n-1} At^i = \frac{A}{2}\sum_{j=1}^{n}(t^j + t^{j-1})^2 \geqslant 0.$$

For $0 \leqslant t \leqslant \frac{A}{2B}$, we have that

$$P(t) \geqslant \frac{A + At^{2n}}{2} + \sum_{j=1}^{n-1} At^{2j} - \sum_{j=1}^{n} Bt^{2j-1} = \frac{A}{2}t^{2n} + \frac{A}{2} - Bt + \sum_{j=1}^{n-1}(A - Bt)t^{2j} \geqslant 0.$$

Hence, $P(t) \geqslant 0$ for all $t \leqslant \frac{A}{2B}$, which proves the Lemma as $a_0, a_{2n} > 0$. □

In particular, $t \in \mathcal{G}(a_0, \ldots, a_{2n}) \subseteq \mathcal{F}(a_0, \ldots, a_{2n})$ for all $t \leqslant 0$ and separated sequences $a_0, \ldots, a_{2n}$, and $\mathcal{F}(a_0, \ldots, a_{2n}) = \mathcal{G}(a_0, \ldots, a_{2n}) = \mathbb{R}$ if in addition we have that $a_0, \ldots, a_{2n} > 0$.

Lemma 1.2. If $0 \leqslant m \leqslant n$, then $\mathcal{G}(a_0, \ldots, a_{2m}) \cap \mathcal{G}(a_{2m}, \ldots, a_{2n}) \subseteq \mathcal{G}(a_0, \ldots, a_{2n})$ for all sequences $a_0, \ldots, a_{2n}$.

Proof. If $m = 0$ or $m = n$, then the Lemma is clear as $\mathcal{G}(a_{2m}, \ldots, a_{2n}) = \mathcal{G}(a_0, \ldots, a_{2n})$ or $\mathcal{G}(a_0, \ldots, a_{2m}) = \mathcal{G}(a_0, \ldots, a_{2n})$. Otherwise, if $t \in \mathcal{G}(a_0, \ldots, a_{2m})$ and $t \in \mathcal{G}(a_{2m}, \ldots, a_{2n})$, then

$$\frac{a_0 + a_{2n}t^{2n}}{2} + \sum_{i=1}^{2n-1} a_i t^i = \left(\frac{a_0 + a_{2m}t^{2m}}{2} + \sum_{i=1}^{2m-1} a_i t^i\right) + t^{2m}\left(\frac{a_{2m} + a_{2n}t^{2(n-m)}}{2} + \sum_{i=2m+1}^{2n-1} a_i t^{i-2m}\right)$$

is nonnegative, so $t \in \mathcal{G}(a_0, \ldots, a_{2n})$ as well. □

In particular, $\mathcal{G}(a_0, \ldots, a_{2m}) \subseteq \mathcal{G}(a_0, \ldots, a_{2n})$ if $\mathcal{G}(a_{2m}, \ldots, a_{2n}) = \mathbb{R}$.

Lemma 1.3. If n is odd and $a_0, \ldots, a_{2n}$ is a separated sequence such that $a_i \geqslant C$ when $i \neq n$ and $(2n-1)C + a_n \geqslant 0$ for some $C > 0$, then $\mathcal{G}(a_0, \ldots, a_{2n}) = \mathbb{R}$.

Proof. Since $a_0, \ldots, a_{2n}$ is separated, $t \in \mathcal{G}(a_0, \ldots, a_{2n})$ for all $t \leqslant 0$. For $t > 0$, we have that

$$\frac{a_0 + a_{2n}t^{2n}}{2} + \sum_{i=1}^{2n-1} a_i t^i \geqslant \frac{C + Ct^{2n}}{2} + \sum_{i=1}^{n-1}(Ct^i + Ct^{2n-i}) + a_n t^n \geqslant ((2n-1)C + a_n)t^n \geqslant 0$$

since $Ct^i + Ct^{2m-i} \geqslant 2Ct^n$ for all i, so $t \in \mathcal{G}(a_0, \ldots, a_{2n})$ as well and $\mathcal{G}(a_0, \ldots, a_{2n}) = \mathbb{R}$. □

We now demonstrate a strategy for Tam to stop the game. The strategy is divided into stages, where in each stage Tam works towards an intermediate game state. In this way, Tam achieves a sequence of game states that ends with the game being stopped. In what follows, the game may stop before all intermediate states are reached. Since Tam's goal is to stop the game, this does not impact her strategy.

Note that 0 will never be written on the board since Tim initially only writes nonzero numbers. We now give the definition of each state, followed by the strategy to achieve it starting from the previous state.

State 1. The board has of an odd number of reals, with more positive reals than negative reals.

Strategy. Throughout this strategy, we assume that State 1 has not yet been achieved.

If there are an even number of reals on the board, all of which are negative, Tam chooses $P(x)$ arbitrarily.

If there are an odd number $2n + 1$ of reals on the board, Tam chooses $P(x) = \sum_{i=0}^{2n} a_i x^i$ so that $-a_0, \ldots, -a_{2n}$ is separated. Lemma 1.1 then gives that $P(r) < 0$ for all $r \leqslant 0$.

If there are an even number $2n + 2$ of reals on the board, at least one of which is positive, Tam chooses $P(x) = \sum_{i=0}^{2n+1} a_i x^i$ so that $-a_0, \ldots, -a_{2n}$ is separated and $a_{2n+1} > 0$. Lemma 1.1 then gives that $P(r) = a_{2n+1}r^{2n+1} + \sum_{i=0}^{2n} a_i r^i < 0$ for all $r \leqslant 0$.

In the latter two cases, Tim only writes positive numbers on the board. Choosing $P(x)$ in this way repeatedly, the first case happens at most once and Tam achieves State 1. □

State 2. The numbers on the board can be ordered as $a_0, \ldots, a_{2n_0}, \ldots, a_{2n_1}, \ldots, a_{2n_\ell}$ so that $a_0, \ldots, a_{2n_0}$ is a separated sequence of positive reals and $a_{2n_{j-1}}, \ldots, a_{2n_j}$ satisfies the hypotheses of Lemma 1.3 for each $1 \leqslant j \leqslant \ell$.

Strategy. If all numbers on the board are positive, then State 2 has already been achieved with $\ell = 0$. Otherwise, let $C = \frac{A}{2B}$, where $-B < 0$ is the smallest number on the board and $A > 0$ is such that the numbers on the board can be ordered into an A-separated sequence. If Tam chooses $P(x)$ in each round so that only positive numbers are added to the board and numbers greater than C are added to the board infinitely often, then the game is eventually in State 2.

If there are an even number $2n+2$ of reals on the board, Tam chooses $P(x) = \sum_{i=0}^{2n+1} a_i x^i$ so that $a_0, \ldots, a_{2n}$ is separated and $a_{2n+1} < 0$. Lemma 1.1 gives that $P(r) = a_{2n+1} r^{2n+1} + \sum_{i=0}^{2n} a_i r^i > 0$ for all $r \leqslant 0$. The number added to the board must then be positive.

If there are an odd number $2n+1$ of reals on the board, Tam chooses $P(x) = \sum_{i=0}^{2n} a_i x^i$ so that $a_0, \ldots, a_{2n}$ is separated, $a_0, \ldots, a_{2m}$ is A-separated, $a_0, \ldots, a_{2m} \geqslant -B$, and $a_{2m}, \ldots, a_{2n} > 0$ for some $0 \leqslant m \leqslant n$. This is possible initially by construction, and it always remains possible as only positive numbers are ever added to the board. Lemma 1.1 gives that $t \in \mathcal{G}(a_0, \ldots, a_{2m})$ for all $t \leqslant C$ and $\mathcal{G}(a_{2m}, \ldots, a_{2n}) = \mathbb{R}$, so $t \in \mathcal{G}(a_0, \ldots, a_{2n}) \subseteq \mathcal{F}(a_0, \ldots, a_{2n})$ for all $t \leqslant C$ by Lemma 1.2. The number added to the board must then be greater than C.

Choosing $P(x)$ in this way repeatedly, Tam achieves State 2. □

State 3. The game is stopped.

Strategy. Suppose that the game is in State 2 with the ordering $a_0, \ldots, a_{2n_0}, \ldots, a_{2n_1}, \ldots, a_{2n_\ell}$. By Lemmas 1.1 and 1.3, we have that $\mathcal{G}(a_0, \ldots, a_{2n_0}) = \mathbb{R}$ and $\mathcal{G}(a_{2n_{j-1}}, \ldots, a_{2n_j}) = \mathbb{R}$ for each $1 \leqslant j \leqslant \ell$. Repeatedly applying Lemma 1.2 then gives $\mathcal{G}(a_0, \ldots, a_{2n_\ell}) = \mathbb{R}$. Since $a_0, a_{2n_\ell} > 0$, $\mathcal{F}(a_0, \ldots, a_{2n_\ell}) = \mathbb{R}$ as well, so Tam can choose $P(x) = \sum_{i=0}^{2n_\ell} a_i x^i$ to stop the game. □

Following this strategy, Tam can stop the game regardless of Tim's choices, so Tim cannot ensure that the game is never stopped.

Solution 2. We present another way for Tam to execute steps (1) and (2) in the strategy outline from the common remarks without using separated sequences. In Solution 1, the polynomials used have separated sequences of coefficients, which allows for a positive lower bound on their real roots. In this solution, we weaken this to only having alternating sequences of coefficients and instead obtain big positive numbers on the board by choosing the constant coefficient of $P(x)$ to be large.

The edge case of the board only having numbers of one sign can be handled as in Solution 1. Henceforth, we assume that the board contains numbers of both signs.

Say that a sequence $b_0, \ldots, b_k$ of nonzero reals is *alternating* if $b_0 > 0$, $b_i \geqslant b_{i+1}$ for even $0 \leqslant i < k$, $b_i \leqslant b_{i+1}$ for odd $0 \leqslant i < k$, and $(-1)^k b_k > 0$.

Lemma 2.1. Let $b_0, \ldots, b_k$ be an alternating sequence, and let $P(x) = \sum_{i=0}^{k} b_i x^i$. Then $P(t) > 0$ for all $t \leqslant 0$. In particular, all real roots of $P(x)$ are positive.

Proof. We prove the stronger statement that $Q(x) = \frac{b_0}{2} + \sum_{i=1}^{k} b_i x^i$ satisfies $Q(t) > 0$ for all $r \leqslant 0$. We proceed with induction on k, with the base cases of $k = 0$ and $k = 1$ clear. For the inductive step, if $b_1 < 0$, then the sequence $-b_1, \ldots, -b_k$ is alternating. We thus have for $t \leqslant 0$ that

$$Q(t) = \frac{b_0 + b_1 t}{2} + t\left(\frac{b_1}{2} + \sum_{i=1}^{k-1} b_{i+1} t^i\right) > t\left(\frac{b_1}{2} + \sum_{i=1}^{k-1} b_{i+1} t^i\right) > 0.$$

If $b_1 > 0$ then $b_0 \geqslant b_1$ and $b_2 \geqslant b_1 > 0$, so $k \geqslant 2$ and the sequence $b_2, \ldots, b_k$ is alternating. We thus have for $t \leqslant 0$ that

$$Q(t) = \frac{b_0 - b_1}{2} + \frac{b_2 - b_1}{2} t^2 + \frac{b_1}{2}(t+1)^2 + t^2\left(\frac{b_2}{2} + \sum_{i=1}^{k-2} b_{i+2} t^i\right) > 0.$$

The induction is now complete, and for such sequences all real roots of $P(x)$ are positive. □

Since the numbers on the board do not all have the same sign, they can always be arranged into an alternating sequence. Hence, Tam can always eventually force Tim to only write positive numbers on the board by Lemma 2.1. This concludes step (1).

Say that a sequence $b_0, \ldots, b_k$ of nonzero reals is *top-heavy* if $|b_0| \geqslant |b_1|, \ldots, |b_k|$.

Lemma 2.2. Let $b_0, \ldots, b_k$ be a top-heavy sequence, and let $P(x) = \sum_{i=0}^{k} b_i x^i$. Then $P(t) \neq 0$ for all $|t| \leqslant \frac{1}{2}$. In particular, all real roots of $P(x)$ have magnitude greater than $\frac{1}{2}$.

Proof. By the triangle inequality,

$$|P(t)| = \left|\sum_{i=0}^{k} b_i t^i\right| \geqslant |b_0| - \sum_{i=1}^{k} |b_i| \cdot |t|^i \geqslant |b_0| - \sum_{i=1}^{k} \frac{|b_0|}{2^i} = \frac{|b_0|}{2^k} > 0$$

for all $|t| \leqslant \frac{1}{2}$. □

Suppose that there are an even number of reals on the board. Since they do not all have the same sign, Tam can always arrange the numbers on the board into a sequence $a_0, \ldots, a_k$ such that either $a_0, \ldots, a_k$ or $-a_0, \ldots, -a_k$ is both alternating and top-heavy, depending on the sign of the number of the largest magnitude on the board. Thus, Tam can always eventually force Tim to write numbers greater than $\frac{1}{2}$ on every other turn by Lemmas 2.1 and 2.2. This concludes step (2). We remark here that Solution 1 executes step (2) when the number of reals on the board is odd, while this solution does so when it is even.

Step (3) can be executed as in Solution 1. Note that the notion of separated sequences is not needed for this. Indeed, in the sequence $a_0, \ldots, a_{2n_0}, \ldots, a_{2n_1}, \ldots, a_{2n_\ell}$ used in Solution 1, we may instead impose that the subsequences $a_0, \ldots, a_{2n_0}$ and $a_{2n_{i-1}}, \ldots, a_{2n_i}$ are alternating rather than separated and the proof works in the same way.

Comment. The original proposal did not stipulate that the numbers Tim initially writes on the board are nonzero. This does not significantly impact Tam's strategy, as she may always set the leading coefficients of $P(x)$ to be 0 so the 0's are effectively not on the board. Since this introduces a relatively uninteresting complication to the proof without changing the core idea of the solution, the Problem Selection Committee considers the current formulation to be more suitable for the competition.

Comment. This game can also be played with nonzero integers rather than nonzero reals for the coefficients and roots of $P(x)$. This variant of the problem is significantly easier and admits a wider variety of winning strategies for Tam. We briefly sketch two such strategies below:

- Tam labels Tim's initial numbers as $b_0, \ldots, b_\ell$ in an arbitrary order. In each round, she chooses $P(x) = \sum_{i=0}^{k} b_i x^{k-i}$ and defines b_{k+1} to be the integer root of $P(x)$ that Tim chooses. Hence, $P_{k+1}(x) = xP(x) + b_{k+1}$ for all $k \geqslant \ell$. Since $0 = P_{k+1}(b_{k+2}) = b_{k+2}P(b_{k+2}) + b_{k+1}$, we have that $b_{k+2} \mid b_{k+1}$ for all $k \geqslant \ell$, so there exists a positive integer b with $|b_k| = b$ for all large k. It cannot be the case that $P(b_{k+2}) = P_{k+1}(b_{k+2}) = 0$, so $b_k + b_{k+1} = 0$ and $b_k = b_{k+2}$ for all large k. For such k, we have that $0 = P_{k+1}(b_{k+2}) = P_{k+1}(b_k) = b_k^2 P_{k-1}(b_k) + b_k^2 + b_{k+1} = b_k^2 + b_{k+1}$, so $b_k^2 = -b_{k+1} = b_k \implies b_k = 1$ for all large k. This contradicts $b_k + b_{k+1} = 0$, so the game must stop at some point.

- Tam labels Tim's initial numbers as $b_0, \ldots, b_k$ so that $|b_0| \leqslant \cdots \leqslant |b_k|$, i.e. she sorts the numbers by magnitude. She first chooses $P(x) = \sum_{i=0}^{k} b_i x^i$. For $|r| \geqslant 2$, we have by the triangle inequality that $|P(r)| \geqslant |b_k| r^k - \sum_{i=0}^{k-1} |b_i| r^i \geqslant |b_k|(r^k - \sum_{i=0}^{k-1} r^i) > 0$, so $P(x)$ has no integer roots with magnitude greater than 1. If the game does not stop, then Tim writes b on the board with $|b| = 1$. Tam then chooses $P_{k+1}(x) = b + \sum_{i=1}^{k+1} b_{i-1} x^i = xP(x) + b$. Since $|b| \leqslant |b_0| \leqslant \cdots \leqslant |b_k|$, the previous argument implies that $P_{k+1}(x)$ has no integer roots with magnitude greater than 1. If the game does not stop, then Tim writes c on the board with $|c| = 1$. This gives $0 \equiv P_{k+1}(c) \equiv P_{k+1}(1) \equiv P(1) + 1 \equiv P(b) + 1 \equiv 1 \pmod 2$, a contradiction.

Combinatorics

C1. Let $n \geqslant 3$ be an integer, and let S_n be the set of points (x, y) in the plane such that x and y are nonnegative integers and $x + y < n$. A line in the plane is called *interesting* if it is not parallel to the x-axis, the y-axis, or the line $x + y = 0$.

Determine all nonnegative integers k such that there exist n lines satisfying both of the following:

- the union of the n lines contains every point of S_n, and
- exactly k of the lines are interesting.

(U.S.A.)

Answer: For any value of n, the only possible values of k are $0, 1$ and 3.

Solution. Define a *boring* line to be a line that is not interesting. Say that the pair (n, k) is *good* if there exist n lines, exactly k of which are interesting, whose union contains S_n. Let T be the triangle formed by the lines $x = 0$, $y = 0$, and $x + y = n - 1$.

Lemma. If $n \geqslant 4$, then (n, k) is good if and only if $(n - 1, k)$ is good.

Proof. If $(n - 1, k)$ is good, then consider $n - 1$ lines that cover S_{n-1}, exactly k of which are interesting. Add the boring line $x + y = n - 1$. These n lines cover S_n and exactly k of them are interesting. This shows that (n, k) is good.

Conversely, suppose (n, k) is good and $n \geqslant 4$. The triangle T contains $3n - 3 > 2n$ points of S_n on its boundary. If n lines cover S_n, k of which are interesting, then by the pigeonhole principle, one of them intersects the boundary of T in at least three points. Call this line ℓ. Then ℓ is a side of T, so ℓ is a boring line. The remaining $n - 1$ lines cover S_n minus the points on ℓ, and exactly k of them are interesting. But S_n minus the points on ℓ is just a translated copy of S_{n-1}, so $(n - 1, k)$ is good. □

By the lemma, the possible values of k do not depend on n, so it suffices to consider only the case $n = 3$. The constructions for $k = 0, 1, 3$ are described below and illustrated in Figure 1.

1. $k = 0$ is obtained with the lines $y = 0$, $y = 1$, and $y = 2$.
2. $k = 1$ is obtained with the lines $y = 0$, $y = 1$, and any interesting line through the point $(0, 2)$.
3. $k = 3$ is obtained with the lines $x = y$, $2x + y = 2$, and $x + 2y = 2$.

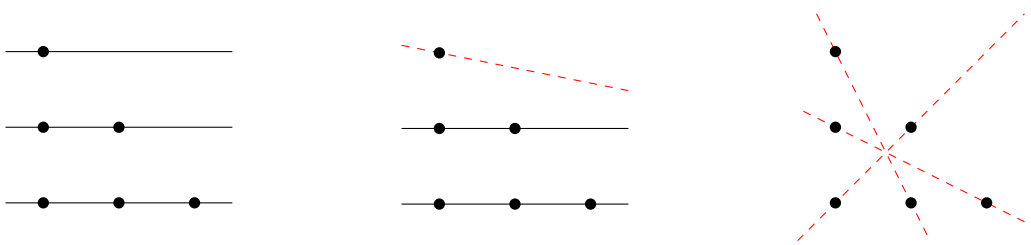


Figure 1: Constructions for $n = 3$ with $k = 0, 1, 3$, respectively. Interesting lines are dashed and coloured red.

Finally, we show that when $n = 3$ we cannot have $k = 2$. Suppose for the sake of contradiction that three lines cover S_3, exactly two of which are interesting. We consider whether one of the lines contains three points or not.

Case 1: There is a line containing three points.

Then this line is a side of T, which is boring. The remaining points of S_3 not covered by this line is a translation of S_2, which is covered by the remaining two interesting lines. This is not possible, since any line through two points in S_2 is boring.

Case 2: There is no line containing three points.

Then each line contains exactly two points of S_3. But there are only three interesting lines that contain exactly two points of S_3, namely, the lines $x = y$, $2x + y = 2$, and $x + 2y = 2$. It is not possible to cover S_3 using two of these three interesting lines and a boring line. Therefore $k = 2$ is not possible.

C2. There is a row of n paddocks, labelled from left to right with the integers 1 to n, where $n \geqslant 3$. Skippy the Kangaroo grazes in the paddocks according to the following rule:

> If Skippy is grazing in paddock k, then she makes a sequence of k hops from a paddock to an adjacent paddock. The first of the k hops is always to the right, unless Skippy is in paddock n, in which case it is to the left. Each of the following $k-1$ hops is in the same direction as the previous hop, unless Skippy is in paddock 1 or n. Skippy grazes in the paddock that she is in after the k^{th} hop.

For example, if $n = 8$ and Skippy is grazing in paddock 3, then the next four paddocks she grazes in are 6, 4, 8 and 2, in this order.

Skippy continues grazing in this way indefinitely. A paddock is *overgrazed* if Skippy eventually grazes in it, regardless of the paddock that she starts in. Determine all $n \geqslant 3$ for which there is exactly one overgrazed paddock.

(Colombia)

Answer: There is exactly one overgrazed paddock if and only if $n = 3 \cdot 2^s$ for some nonnegative integer s.

Common remarks. Write $[n]$ for $\{1, 2, \ldots, n\}$. We will say that n is *convergent* if there is exactly one overgrazed paddock.

Suppose that Skippy is grazing in paddock k. Then:

- if $k \leqslant n/2$ she makes k hops to the right and grazes in paddock $2k$ next;
- if $n/2 < k \leqslant n-1$ she makes $n-k$ hops to the right followed by $k-(n-k) = 2k-n$ hops to the left, and grazes in paddock $n-(2k-n) = 2(n-k)$ next; and
- if she is grazing in paddock n she makes $n-1$ hops to the left followed by one hop to the right, and grazes in paddock 2 next.

Thus if $f : [n] \to [n]$ is the function such that Skippy grazes in paddock $f(k)$ next after grazing in paddock k, then

$$f(k) = \begin{cases} 2k & \text{if } k \leqslant n/2, \\ 2(n-k) & \text{if } n/2 < k \leqslant n-1, \\ 2 & \text{if } k = n. \end{cases} \qquad (*)$$

If Skippy starts in paddock a, then the sequence of paddocks she grazes in is given by the orbit of a under f; that is, it is given by the sequence $a, f(a), f^2(a), \ldots$. Since there are only finitely many possible values for $f^t(a)$, the orbit of any point under f is eventually periodic. It follows that there exists an overgrazed paddock if and only if f has a unique periodic orbit. Moreover, if f has a unique periodic orbit then every integer m belonging to this orbit is overgrazed, so there is a unique overgrazed paddock if and only if the unique periodic orbit of f is a fixed point. Thus $n \geqslant 3$ is convergent if and only if f has a unique periodic point.

Solution 1. The key to this solution is Lemma 1:

Lemma 1. Let $n = 2^s x$ where x is odd, and suppose that $k < n$. If 2^t divides k for $t \leqslant s$, then 2^{t+1} divides $f(k)$.

Proof. Write $k = 2^t y$ for some integer y. If $k \leqslant n/2$ then $f(k) = 2k = 2^{t+1}y$, so 2^{t+1} divides $f(k)$. On the other hand, if $k > n/2$ then $f(k) = 2(n-k) = 2(2^s x - 2^t y) = 2^{t+1}(2^{s-t}x - y)$, so 2^{t+1} divides $f(k)$ in this case too. □

Claim 1.1. The function f has a fixed point if and only if 3 divides n, in which case $q = 2n/3$ is the unique fixed point of f.

Proof. Suppose that q is a fixed point of f. Then $q > n/2$, because $2q \neq q$, and $q \neq n$, because $f(n) = 2 \neq n$. Hence $n/2 < q \leqslant n-1$, and q satisfies $f(q) = 2(n-q) = q$. This rearranges to $q = 2n/3$, so 3 must divide n and any fixed point of f is unique. For $n = 3m$ we have $q = 2m$, and checking we see that $f(q) = q$ does indeed hold. □

From now on we assume that 3 divides n, and set $q = 2n/3$.

Claim 1.2. Suppose that $n = 3 \cdot 2^s$. If $a \in [n]$ then the orbit of a contains q, so n is convergent.

Proof. Write $a = 2^t y$ where y is odd and $t \leqslant s+1$, because $a < 3 \cdot 2^s$. If $t < s$ then by Lemma 1 2^{t+1} divides $f(a)$, and inductively we conclude that a has an iterate $a' = 2^s z$ with $z \in \{1, 2, 3\}$ (since $n = 3 \cdot 2^s$). If $z = 1$ or 2 then either a' or $f(a')$ is equal to q and we are done. Otherwise, $f(a') = f(n) = 2$, and then $f^{s+1}(a) = f^s(2) = 2^{s+1} = q$. □

Claim 1.3. Suppose that n is not of the form $3 \cdot 2^s$. Then there exists an orbit of f that does not contain q, so n is not convergent.

Proof 1. Write $n = 3 \cdot 2^s x$ where $x > 1$ is odd. Then $q = 2^{s+1}x$, and if $f(k) = q$ then either $2k = q$, and so $k = q/2 = 2^s x$; or $2(n-k) = q$, and so $k = n - q/2 = q$. Thus if $a \neq q$ then the orbit of a contains q if and only if it contains $q/2 = 2^s x$.

Consider the point $a = 2^{s+1} < n$. By Lemma 1 with $t = s$ every iterate of a is divisible by 2^{s+1}. It follows that the orbit of a does not contain $q/2$ and hence does not contain q. □

Proof 2. Since n is not of the form $3 \cdot 2^s$, q has an odd prime factor p. Let $k \in [n]$ be such that p does not divide k. Then $k \neq n$, because p divides n, so $f(k) = 2k$ or $f(k) = 2n - 2k$. Since p divides n but not 2 or k we conclude that p does not divide $f(k)$. Since p divides the fixed point q, it follows that the orbit of 1 does not contain q. □

Solution 2. In this second solution we use Lemma 2 below to reduce to the case where $n = 2$ or n is odd. For each $n \geqslant 2$ we define f_n to be the function f of equation $(*)$ with the given value of n, and we write

$$E_n = \{\ell \in [n] : \ell \text{ is even}\}.$$

Lemma 2. If $k < n$ then $f_{2n}(2k) = 2f_n(k)$.

Proof. If $k \leqslant n/2$ then $f_{2n}(2k) = 2(2k) = 2f_n(k)$; and if $n/2 < k$ then $f_{2n}(2k) = 2(2n-2k) = 2 \cdot 2(n-k) = 2f_n(k)$. □

Claim 2.1. Let $n = 2m \geqslant 4$, and let $O = (a_1, a_2, \ldots, a_r)$ be a periodic orbit of f_m. If O does not pass through m let

$$O' = (2a_1, 2a_2, \ldots, 2a_r);$$

otherwise, if O does pass through m with $a_r = m$, let

$$O' = (2, 2a_1, 2a_2, \ldots, 2a_r).$$

Then O' is a periodic orbit of f_n. Moreover, the map $O \mapsto O'$ is a bijection between the periodic orbits of f_m and the periodic orbits of f_n.

Proof. When O does not pass through m, the fact that O' is a periodic orbit of f_n follows immediately from Lemma 2. So suppose that $a_r = m$, and write $O' = (a'_0, a'_1, a'_2, \ldots, a'_r) = (2, 2a_1, 2a_2, \ldots, 2a_r)$. We have $a_1 = f_m(m) = 2$ and $a_2 = f(2) = 4$, so $f_n(a'_r) = f_n(n) = 2 = a'_0$ and $f(a'_0) = f(2) = 4 = 2a_1 = a'_1$. For the remaining entries the equality $f_n(a'_i) = a'_{i+1}$ holds by Lemma 2, so O' is a periodic orbit of f_n.

Using the fact that O' passes through n if and only if O passes through m we see that the map $O \mapsto O'$ is one-to-one on the set of periodic orbits of f_m. Now let O' be a periodic orbit of f_n. By $(*)$ $f(k)$ is always even, so the periodic orbits of f_n must all lie in E_n. Dividing each entry of O' by 2 and deleting 1 if it appears we obtain a periodic orbit O of f_m mapping to O'. □

Let $n = 2m \geqslant 4$. By Claim 2.1 f_n has a unique periodic orbit if and only if f_m does; and if q is a fixed point of f_m then $2q$ is a fixed point of f_n unless $q = m$. It follows that $n = 2m$ is convergent if and only if m is convergent and the unique periodic point of f_m is not m.

It suffices to consider the cases where $n = 2$ or n is odd.

- For $n = 2$ it is easily seen that (2) is the unique periodic orbit of f_2, so no power of 2 greater than 3 is convergent.
- For $n = 3$ it is easily seen that (2) is the unique periodic orbit of f_3, so every n of the form $3 \cdot 2^s$ is convergent.
- For odd $n \geqslant 5$, Claim 2.2 below implies that every element of E_n is periodic, so n is not convergent because $|E_n| \geqslant 2$.

Claim 2.2. If n is odd then the restriction of f_n to E_n is a bijection $E_n \to E_n$.

Proof. The restriction of f_n to E_n maps E_n into itself because $f(k)$ is always even. It suffices to show that it is surjective, so let $a = 2b \in E_n$. If $b = 2c$ is even then $b \in E_n$, and $f_n(b) = a$. Otherwise, if b is odd let $k = n - b = n - a/2 > n - n/2 = n/2$. Then k is even, because n and b are both odd, and $f_n(k) = 2(n - k) = 2b = a$. □

Comment. For $n = 2^s x$ with x odd define $E_n^* = \{\ell \in [n] : 2^{s+1} \text{ divides } \ell\}$. By Lemma 1 with $t = s$ the function f maps E_n^* into E_n^*, and the proof of Claim 2.2 can be adjusted to show that the restriction of f to E_n^* is a bijection. It follows that every element of E_n^* is periodic, and this can be used to give a third proof of Claim 1.3.

C3. There are 2025 white balls and 2025 black balls arranged in a row. A *balanced segment* is a contiguous nonempty segment of balls that contains the same number of white balls as black balls. A *balanced removal* is the operation of selecting a balanced segment S, removing all the balls in S, and shifting left the balls that were to the right of S so that the remaining balls form a contiguous row.

Determine the smallest positive integer N satisfying the following property: for every configuration of balls and for every integer k satisfying $0 \leqslant k \leqslant 2025$, there exists a sequence of at most N balanced removals after which there are exactly $2k$ remaining balls.

(Islamic Republic of Iran)

Answer: The smallest such positive integer is $N = 2$.

Solution 1. We first show that $N = 1$ does not satisfy the property. Consider the configuration

and the integer $k = 1$. There are three contiguous segments consisting of $2 \cdot 2024$ balls, and we observe that none of them are balanced. Therefore we cannot leave exactly 2 balls after performing a single balanced removal.

We now prove that at most $N = 2$ balanced removals are sufficient to leave exactly $2k$ balls for every $0 \leqslant k \leqslant 2025$. We define a *minimal balanced segment* to be a balanced segment that cannot be divided into a balanced segment followed by another balanced segment.

Lemma. Consider a minimal balanced segment S consisting of $2m$ balls. For every integer $1 \leqslant l \leqslant m$ there exists a balanced segment of length $2l$ contained in S.

Proof. For each integer $0 \leqslant i \leqslant 2m$, let a_i be the number of black balls minus the number of white balls among the first i balls of S. Then $|a_i - a_{i+1}| = 1$ for $0 \leqslant i \leqslant 2m-1$, and moreover $a_i \neq 0$ for all $1 \leqslant i \leqslant 2m-1$ by assumption. Hence the a_i's for $1 \leqslant i \leqslant 2m-1$ are either all positive or all negative.

For each integer $0 \leqslant i \leqslant 2(m-l)$, let b_i be the number of black balls in the segment starting with the $(i+1)^{\text{st}}$ ball and ending with the $(i+2l)^{\text{th}}$ ball of S. Then $|b_i - b_{i+1}| \leqslant 1$ for all integers $0 \leqslant i \leqslant 2(m-l)-1$. We also compute that $b_0 = l + a_{2l}/2$ and $b_{2(m-l)} = l - a_{2(m-l)}/2$. Since a_{2l} and $a_{2(m-l)}$ have the same sign, either $b_0 \geqslant l \geqslant b_{2(m-l)}$ or $b_0 \leqslant l \leqslant b_{2(m-l)}$. In either case, there exists an integer $0 \leqslant i \leqslant 2(m-l)$ satisfying $b_i = l$, as desired. □

Given the row of 4050 balls, we decompose it into minimal balanced segments. Let

$$0 = s_0 < s_1 < \cdots < s_p = 2025$$

be the integers for which the segment starting with the $(2s_i+1)^{\text{st}}$ ball and ending with the $(2s_{i+1})^{\text{th}}$ ball is a minimal balanced segment for every $0 \leqslant i \leqslant p-1$. Given the integer $0 \leqslant k \leqslant 2025$, if $s_i = 2025 - k$ for some $0 \leqslant i \leqslant p$, then we may simply remove the first $2(2025-k)$ balls, which form a balanced segment. Otherwise, there exists an index $0 \leqslant i \leqslant p-1$ with the property that

$$s_i < 2025 - k < s_{i+1}.$$

We now apply the lemma to the minimal balanced segment S starting with the $(2s_i+1)^{\text{st}}$ ball and ending with the $(2s_{i+1})^{\text{th}}$ ball. There exists a balanced segment S_1 of length $2(2025-k-s_i)$ within S, and also the first $2s_i$ balls form a balanced segment S_2 disjoint from S. Thus we may leave $2k$ balls by removing S_1 and S_2.

Solution 2. We provide a second proof of the lemma. The rest of the solution remains the same.

Proof of Lemma. As in the first proof, for $0 \leqslant i \leqslant 2m$ define a_i to be the number of black balls minus the number of white balls among the first i balls of S. We prove the more general statement that if S is balanced and $a_i \geqslant 0$ for all i, then there exists a balanced segment in S of length $2l$ for every $1 \leqslant l \leqslant m$.

We induct on m. When $m = 1$, we have $l = 1$ and hence the statement clearly holds. Assume now that $m \geqslant 2$ and the statement holds for smaller values of m. We note that $a_1 = a_{2m-1} = 1$, and hence the first ball is black and the last ball is white.

Case 1: $a_i \geqslant 1$ for all $1 \leqslant i \leqslant 2m-1$.

The segment T starting with the 2nd ball and ending with the $(2m-1)^{\text{st}}$ ball is a balanced segment of length $2m-2$, satisfying $a_i - 1 \geqslant 0$ for all $1 \leqslant i \leqslant 2m-1$. By the inductive hypothesis, for each $1 \leqslant l \leqslant m-1$ there exists a balanced segment of length $2l$ within T. For $l = m$, we note that S itself is a balanced segment of length $2m$.

Case 2: $a_i = 0$ for some $1 \leqslant i \leqslant 2m-1$.

Without loss of generality, we may assume that $i \leqslant m$. (If not, we can flip the segment S, then replace black balls with white balls and vice versa.) If $2l \leqslant 2m - i$, we may apply the inductive hypothesis to the balanced segment consisting of the last $2m - i$ balls. In the case when $2l > 2m - i$, we note that the i^{th} ball is white and $(i+1)^{\text{st}}$ ball is black because $a_{i-1} = a_{i+1} = 1$. Consider the sequence of balls T we obtain by removing the i^{th} and $(i+1)^{\text{st}}$ ball from S. By applying the inductive hypothesis to T, we obtain a balanced segment $T' \subseteq T$ of length $2(l-1)$, noting that $2l > m \geqslant 2$ and thus $l \geqslant 2$. Let us say that T' starts with the x^{th} ball and ends with the $(x+2l-3)^{\text{rd}}$ ball. Then $x + 2l - 3 \leqslant 2m - 2$ implies

$$x \leqslant 2m + 1 - 2l \leqslant 2m + 1 - (2m - i + 1) = i$$

and similarly $x \geqslant 1$ implies

$$x + 2l - 3 \geqslant 2l - 2 \geqslant 2m - i - 1 \geqslant i - 1.$$

This shows that T' together with the i^{th} and $(i+1)^{\text{st}}$ ball from S form a balanced segment of length $2l$ in S. □

Comment. The lemma can also be deduced from the "mountain climbing" puzzle appearing in the article *Mountain Climbing, Ladder Moving, and the Ring-Width of a Polygon* by Goodman–Pach–Yap. By the Intermediate Value Theorem, there exists a time at which the distance between the climbers equals $2l$. If the coordinates of the climbers are integral, we are done. Otherwise, one shows that the slopes the climbers are on must be parallel to each other, using the fact that $2l$ is an even integer. Then we may simultaneously decrease the altitude of the climbers until both have integral coordinates.

C4. Bluey is placing lollies on a 1000×1000 grid. Initially, every cell of the grid is empty. At each step, Bluey chooses a row or column in which the total number of lollies is a multiple of 3, chooses an empty cell in this row or column, and places a lolly in it. If there is no such row or column, Bluey stops.

Determine the maximum number of lollies that Bluey can place.

(China)

Answer: The maximum number of lollies that Bluey can place is 3995.

Solution 1. Let n be any integer greater than or equal to 5, then we will show that the answer is $4n-5$ for an $n \times n$ grid. The desired result can be obtained by setting $n = 1000$. First, we show that it is possible to place $4n-5$ lollies. In the below diagram, we number the lollies $1, 2, \ldots, 4n-5$ such that Bluey places the i^{th} lolly on the i^{th} turn.

	3	4		$\ldots$				
$4n-5$	$4n-6$	5	6	$\ldots$				
	$4n-7$	$4n-8$	7	$\ldots$				
		$4n-9$	$4n-10$	$\ldots$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
				$\ldots$	$2n+4$	$2n-5$	$2n-4$	
				$\ldots$	$2n+3$	$2n+2$	$2n-3$	$2n-2$
2	1			$\ldots$		$2n+1$	$2n$	
				$\ldots$			$2n-1$	

To show the upper bound, let N be the total number of lollies that Bluey places. We wish to show that $N \leqslant 4n-5$. For any $i = 1, 2, \ldots, n$ and time $t = 0, 1, \ldots, N$, let $u_i(t) \in \{0, 1, 2\}$ denote the residue modulo 3 of the number of lollies in row i after Bluey has placed the first t lollies. Define $v_i(t)$ similarly for column i. Let $P_0(t), P_1(t)$, and $P_2(t)$ be the number of 0's, 1's, and 2's respectively among the $2n$ values $u_1(t), \ldots, u_n(t), v_1(t), \ldots, v_n(t)$. Then we have $P_0(t) + P_1(t) + P_2(t) = 2n$ for all t, and $(P_0(0), P_1(0), P_2(0)) = (2n, 0, 0)$.

Now suppose that on her t^{th} step, Bluey places a lolly in row i and column j. Then we must have $u_i(t) = 0$ or $v_j(t) = 0$. We classify each of Bluey's moves as follows:

- An *A-move* is one where $u_i(t) = v_j(t) = 0$. An A-move updates $u_i(t+1) = v_j(t+1) = 1$, thus we have $(P_0(t+1), P_1(t+1), P_2(t+1)) = (P_0(t) - 2, P_1(t) + 2, P_2(t))$.
- A *B-move* is one where $\{u_i(t), v_j(t)\} = \{0, 1\}$. A B-move updates $\{u_i(t+1), v_j(t+1)\} = \{1, 2\}$, thus we have $(P_0(t+1), P_1(t+1), P_2(t+1)) = (P_0(t) - 1, P_1(t), P_2(t) + 1)$.
- A *C-move* is one where $\{u_i(t), v_j(t)\} = \{0, 2\}$. A C-move updates $\{u_i(t+1), v_j(t+1)\} = \{0, 1\}$, thus we have $(P_0(t+1), P_1(t+1), P_2(t+1)) = (P_0(t), P_1(t) + 1, P_2(t) - 1)$.

Now suppose that Bluey makes a A-moves, b B-moves, and c C-moves. Next, we have:

$$P_0(N) - 2n = P_0(N) - P_0(0) = \sum_{i=0}^{n-1} (P_0(t+1) - P_0(t)) = -2a - b.$$

Similarly, we can show that $P_1(N) = 2a + c$ and $P_2(N) = b - c$. From these, we deduce that:

$$N = a + b + c = a + (2n - 2a - P_0(N)) + (P_1(N) - 2a) = 2n - 3a - P_0(N) + P_1(N).$$

Also, note that $a \geqslant 1$ since Bluey's first move must be an A-move. Bearing this in mind, we finish with some light casework:

- If $a \geqslant 2$, then we have $N \leqslant 2n - 3a + P_1(N) \leqslant 4n - 3a < 4n - 5$, since $P_1(N) \leqslant 2n$ by definition.

- If $a = 1$ and $P_1(N) \leqslant 2n-2$, then we have $N \leqslant 2n-3a+P_1(N) \leqslant 2n-3+2n-2 = 4n-5$.

- If $a = 1$ and $P_1(N) \geqslant 2n - 1$, then at least $2n - 1$ of $u_1(N), \ldots, u_n(N), v_1(N), \ldots, v_n(N)$ must have value 1. Then, since $\sum_{i=1}^{n} u_i(N) \equiv \sum_{i=1}^{n} v_i(N) \pmod 3$, this forces the $(2n)^{\text{th}}$ value to also be 1, so $P_1(N) = 2n$. In this case, Bluey's last move must be an A-move. Since Bluey's first move is also an A-move, this either contradicts our assumption that $a = 1$ or implies that $N = 1 < 4n - 5$.

In all cases, we have proven the desired bound.

Solution 2. We provide an alternate proof that we must have $N \leqslant 4n - 5$. Number the rows and columns $1, 2, \ldots, n$. We call a particular step a *row move* if Bluey adds a lolly to a row where the number of lollies was previously a multiple of 3. For each $i = 1, 2, \ldots, n$, let r_i be the number of row moves made in row i. Similarly define column moves and c_i. (Note that a move can be both a row move and a column move.) Let $R = \sum_{i=1}^{n} r_i$ and $C = \sum_{i=1}^{n} c_i$. Assume without loss of generality that Bluey's final move is a row move.

Firstly, note that every move must contribute at least 1 to $\sum_{i=1}^{n}(r_i + c_i)$. Moreover, the first move will contribute 2 to this sum. It follows that:

$$N + 1 \leqslant R + C.$$

Secondly, let us consider all the lollies that are added to row i. Among these lollies, r_i of them are placed using row moves. Between any two consecutive row moves in row i, there must be at least two "assisting" column moves adding lollies to row i. Moreover, none of these column moves can be Bluey's very first move, thus the total number of assisting column moves is at most $C - 1$. Summing over i implies that:

$$C - 1 \geqslant \sum_{i=1}^{n} 2(r_i - 1) = 2R - 2n.$$

We can apply an analogous argument by looking at the lollies added to each column. However, the number of assisting row moves is now at most $R - 2$, since Bluey's first and last move are both row moves and neither can be an assisting row move. Thus we obtain:

$$R - 2 \geqslant \sum_{i=1}^{n} 2(c_i - 1) = 2C - 2n.$$

Summing the last two inequalities implies that:

$$R + C - 3 \geqslant 2R + 2C - 4n \Leftrightarrow R + C \leqslant 4n - 3 \Rightarrow N \leqslant R + C - 1 \leqslant 4n - 4.$$

For equality to hold, we must have both $C - 1 = 2R - 2n \Leftrightarrow 2R - C = 2n - 1$ and $R - 2 = 2C - 2n \Leftrightarrow 2C - R = 2n - 2$. But now subtracting these implies that $3(R - C) = 1$ which is absurd since R and C are integers. Thus we must have $N < 4n - 4 \Rightarrow N \leqslant 4n - 5$, as desired.

Comment. There are many other constructions attaining $4n-5$ lollies. We present one other example below:

1	2		$2n$			$\ldots$				$2n-2$
	3	4	$2n+1$	$2n+2$		$\ldots$				
		5	6	$2n+3$	$2n+4$	$\ldots$				
			7	8	$2n+5$	$\ldots$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
						$\ldots$	$4n-13$	$4n-12$		
						$\ldots$	$2n-8$	$4n-11$	$4n-10$	
						$\ldots$	$2n-7$	$2n-6$	$4n-9$	$4n-8$
	$4n-6$					$\ldots$		$2n-5$	$2n-4$	$4n-7$
	$4n-5$					$\ldots$			$2n-3$	
						$\ldots$				$2n-1$

C5. Let n be a positive integer, and let $(a_1, a_2, \ldots, a_n)$ be any n-tuple of distinct integers. An *inversion* is a pair (i, j) of integers satisfying $1 \leqslant i < j \leqslant n$ and $a_i > a_j$. An inversion (i, j) is called *odd* if $j - i$ is odd. Let A be the number of all inversions, and let B be the number of odd inversions.

(a) Prove that $A \leqslant 5B$ for all n and all n-tuples $(a_1, a_2, \ldots, a_n)$.

(b) Prove that there exist an n and an n-tuple $(a_1, a_2, \ldots, a_n)$ for which $A > 4.99B$.

(Germany)

Solution. Call an inversion (i, j) *even* if it is not odd.

(a) We classify the inversions (i, j) into five types:

1. the odd inversions;
2. the even inversions (i, j) with $i \equiv j + 2 \pmod 4$ and $a_i > a_{(i+j)/2}$;
3. the even inversions (i, j) with $i \equiv j + 2 \pmod 4$ and $a_i < a_{(i+j)/2}$;
4. the even inversions (i, j) with $i \equiv j \pmod 4$ and $a_i > a_{(i+j)/2-1}$; and
5. the even inversions (i, j) with $i \equiv j \pmod 4$ and $a_i < a_{(i+j)/2-1}$.

Observe that each inversion is of exactly one of these five types. Let $\mathcal{B}_k$ be the set of inversions of Type k and let B_k be the number of inversions of Type k, for $k = 1, \ldots, 5$. Then $B = B_1$ and $A = B_1 + B_2 + B_3 + B_4 + B_5$. To show that $A \leqslant 5B$, it suffices to prove that $B_k \leqslant B_1$ for $k = 2, \ldots, 5$. To that end, we will construct an injective map from $\mathcal{B}_k$ to $\mathcal{B}_1$ for each $k = 2, \ldots, 5$.

(i) For Type 2 inversions, we consider the map $\mathcal{B}_2 \to \mathcal{B}_1$ given by

$$(i, j) \mapsto (i, \tfrac{i+j}{2}).$$

This map is clearly injective, and $(i, \frac{i+j}{2})$ is an odd inversion. Indeed, $a_i > a_{(i+j)/2}$ and $\frac{i+j}{2} - i = \frac{j-i}{2}$ is an odd integer since $i \equiv j + 2 \pmod 4$.

(ii) For Type 3 inversions, we consider the map $\mathcal{B}_3 \to \mathcal{B}_1$ given by

$$(i, j) \mapsto (\tfrac{i+j}{2}, j).$$

This map is clearly injective, and $(\frac{i+j}{2}, j)$ is an odd inversion. Indeed, $a_i > a_j$ since (i, j) is an inversion. Thus, $a_{(i+j)/2} > a_i > a_j$ and $j - \frac{i+j}{2} = \frac{j-i}{2}$ is an odd integer since $i \equiv j + 2 \pmod 4$.

(iii) For Type 4 inversions, we consider the map $\mathcal{B}_4 \to \mathcal{B}_1$ given by

$$(i, j) \mapsto (i, \tfrac{i+j}{2} - 1).$$

Similar to the case for Type 2 inversions, this map is well-defined and is injective.

(iv) For Type 5 inversions, we consider the map $\mathcal{B}_5 \to \mathcal{B}_1$ given by

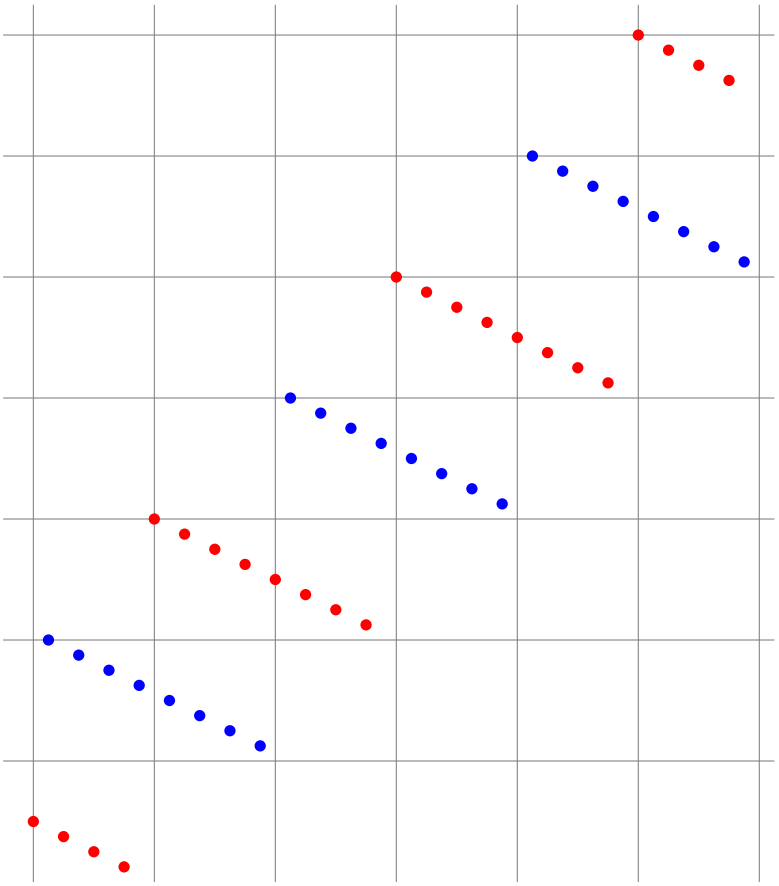
$$(i, j) \mapsto (\tfrac{i+j}{2} - 1, j).$$

Similar to the case for Type 3 inversions, this map is well-defined and is injective.

(b) We will consider n of the form $4mk$ for some positive integers m, k. For $1 \leqslant i \leqslant n$, define a_i as follows.

- If i is even and s is the unique integer such that $4sm < i \leqslant (4s+4)m$, then set $a_i = (6s+2)m - i/2$.
- If i is odd and s is the unique integer such that $(4s-2)m < i < (4s+2)m$, then set $a_i = (6s-1)m - (i+1)/2$.

For example, the figure below depicts the points (i, a_i) for the case $m = 4, k = 3$. The points with i odd are coloured red, while those with i even are coloured blue. The lines are drawn purely for reference.



If i is even then a_i lies in the interval $[4sm, 4sm + 2m)$ and if i is odd then a_i lies in the interval $[4sm - 2m, 4sm)$. From these observations, it is clear that $a_i \neq a_j$ for $i \neq j$.

Letting C be the number of even inversions, we will show that $C > 3.99B$ for some m, k. First, we count the number of even inversions (i, j).

- If i, j are both even, then (i, j) is an inversion if and only if there is some integer s for which $4sm < i < j \leqslant (4s+4)m$. The number of inversions of this form is $k\binom{2m}{2}$.
- If i, j are both odd, then (i, j) is an inversion if and only if there is some integer s for which $(4s-2)m < i < j < (4s+2)m$. By only considering $1 \leqslant s \leqslant k-1$, the number of inversions of this form is at least $(k-1)\binom{2m}{2}$.

In total, we have $C \geqslant (2k-1)\binom{2m}{2}$. Next, we count the number of odd inversions (i, j).

- If i is even and j is odd, then (i, j) is an inversion if and only if there is some integer s for which $4sm < i < j < (4s+2)m$. For each s, the number of such pairs (i, j) is at most $\binom{m}{2}$. Across all s, the number of inversions of this form is at most $k\binom{m}{2}$.

- If i is odd and j is even, then (i, j) is an inversion if and only if there is some integer s for which $(4s+2)m < i < j \leqslant (4s+4)m$. For each s, the number of such pairs (i, j) is at most $\binom{m+1}{2}$. Across all s, the number of inversions of this form is at most $k\binom{m+1}{2}$.

In total, we have $B \leqslant k\binom{m}{2} + k\binom{m+1}{2}$.

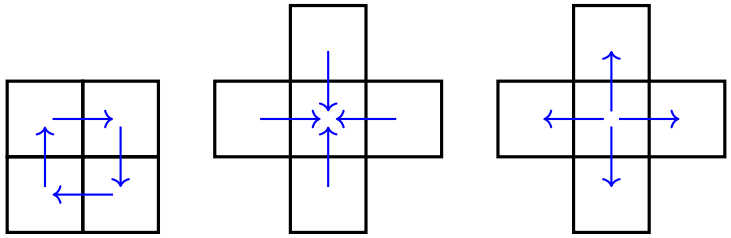
Setting $k = m$, we have $C \geqslant (2m-1)\binom{2m}{2} = 4m^3 - 4m^2 + m$ and $B \leqslant m\binom{m}{2} + m\binom{m+1}{2} = m^3$. Thus, for m sufficiently large, we have $C \geqslant 4m^3 - 4m^2 + m > 3.99m^3 \geqslant 3.99B$.

C6. On a 45×45 square grid there is an echidna. From any cell, the echidna can move to any other cell that shares a side. The echidna makes a sequence of 2024 moves, after which it has visited each cell exactly once. Prove that it is possible to write the numbers from 1 to 2025, one in each cell, in such a way that:

- For any pair of adjacent cells in the same row, the cell containing the larger number was visited earlier.
- For any pair of adjacent cells in the same column, the cell containing the larger number was visited later.

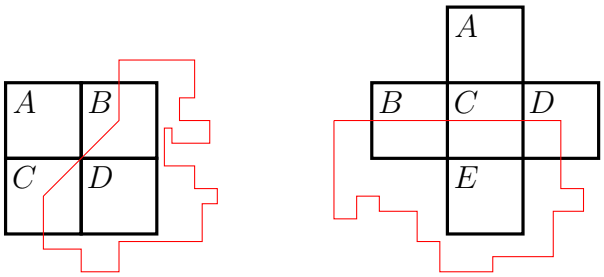
(Spain)

Common remarks. We draw an arrow between each pair of adjacent cells. If the two cells are in the same row, the arrow goes from the cell visited later to the cell visited earlier. If the two cells are in the same column, the arrow goes the other way around. In each case, we want each arrow to point to the smaller number. If the arrows do not form any cycle, then it is possible to write the numbers as required, in a sequence of 2025 steps. On the i^{th} step we follow the arrows from some unnumbered cell until reaching an unnumbered cell which does not point to another unnumbered cell, and write i in that cell. Therefore, it is enough to prove that the guiding arrows do not form any cycle.



(a) (b) (c)

We call a corner in which four cells meet a *node*. We claim that the four arrows around a node cannot be arranged as in figure (a), and the four arrows around a cell cannot be arranged as in (b) or (c) above.



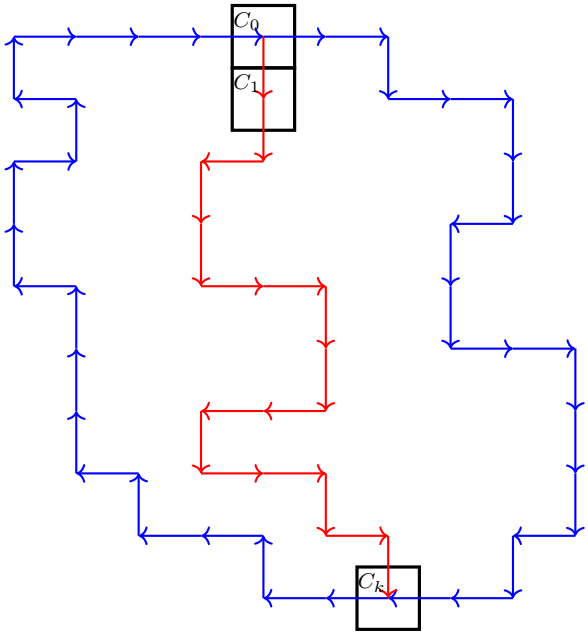
Polygon for (a) Polygon for (b)

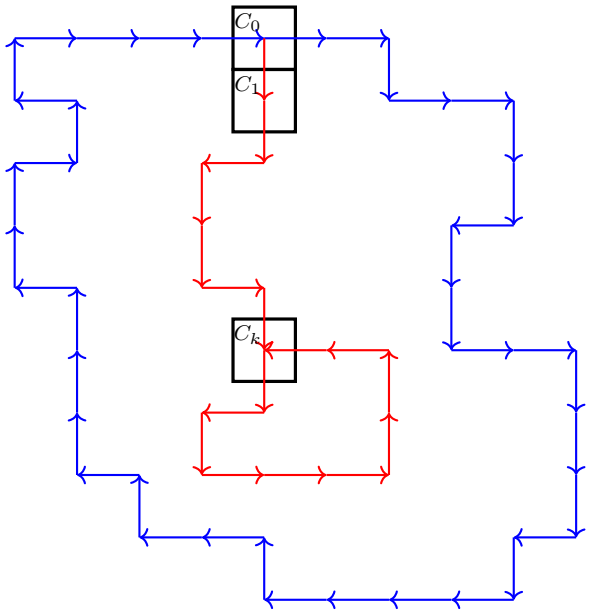
Indeed, by the direction of the arrows, in (a) the echidna visited both B and C before it visited A or D. We can take the path that the echidna took between B and C, and close it with a diagonal segment to form a polygon which separates A and D. But then the path that the echidna takes between A and D must intersect this polygon, which is impossible. Similarly, in (b), the echidna visited both B and D before it visited C, and it visited C before it visited A or E. By taking the echidna's path between B and D, and closing it with C, we create a polygon that separates A and E. The same argument applies for (c).

In the following, we associate a cycle with the closed curve made up of segments connecting the centres of consecutive cells in the cycle.

Solution 1. Suppose that the arrows form a cycle. Consider a cycle L of minimum area, that is, one that encloses the fewest nodes. It cannot enclose a single node, as that would create (a) or its reverse, which are impossible. Therefore there exists a cell C_0 on L and an adjacent cell C_1 such that the arrow between C_0 and C_1 is inside L.

Suppose for the time being that this arrow is oriented from C_0 towards C_1. We define a sequence of cells as follows: as long as C_i is in the interior of L and has not appeared earlier in the sequence, the cell C_i must have four neighbouring cells, and its four arrows cannot be arranged as in (b). Thus there is at least one arrow pointing outwards from C_i, and we let C_{i+1} be the cell it points to. This sequence produces an oriented walk. Let C_k be the last cell in the sequence. Then C_k is either in L or it appears somewhere else in the sequence. In the first case (illustrated on the left below), the walk that we created, together with the segment of L that goes from C_k to C_0, creates a cycle of smaller area than L. In the second case (illustrated on the right below), if $C_k = C_i$, then the walk from C_i to C_k is a cycle that lies entirely in the interior of L, and thus has a smaller area. In either case, we contradict the minimality of L. We conclude that no cycle exists, as we wanted to show.





The case when the arrow between C_0 and C_1 is oriented from C_1 to C_0 is similar. We form a sequence of cells $C_2, \ldots, C_k$ such that there is an arrow pointing from C_{i+1} to C_i for $1 \leqslant i \leqslant k$, and C_k is either on L or a repeat of a cell that appears earlier in the sequence. Either way we have a contradiction analogous to the one we found in the first case.

Solution 2. Suppose that there exists a cycle L. For every node in the interior of L, we will place four charges around it, one in each of the incident cells. Given a node N and a cell C incident to it, consider the two arrows that go from C to other cells incident to N. If both point towards C, or both away from C, we place a positive $(+1)$ charge around N on C. If one arrow points towards C and the other away from C, we place a negative (-1) charge. By double-counting the total charge, we will reach a contradiction.

Start by looking at the charge around each node. Since (a) and its reverse are forbidden, the four charges cannot all be negative. By parity, there is an even number of negative charges around each node. Therefore the sum of all charges around each node is nonnegative, and so is the total sum of all charges.

Next consider the charges on each cell C. If C is in the interior of L, it contains four charges. Since (b) and (c) are impossible, the four charges cannot all be positive. Once again by parity, there is an even number of negative charges on each interior cell, so the total charge on each interior cell is nonpositive. If C is on L, we distinguish the cases in which C is a convex corner, a flat side, or a concave corner (incident to 1, 2 or 3 interior nodes, respectively). In a convex corner, the charge must be negative (-1). In a flat side, the total charge must be 0. In a concave corner, the three charges cannot all be positive, because that would produce (b) or (c). Therefore the total charge is at most $+1$. But L is a polygon with 90° and 270° angles, and therefore has four more convex corners than concave corners. Adding all cells together, we see that the total charge is at most -4, contradicting the node-wise sum. We reach a contradiction, so we conclude that no cycle exists, as we wanted to prove.

C7. Let P be a regular 100-gon. An *Aussie triangulation* is formed by dividing P into a set T of non-overlapping triangles such that:

- there are exactly 2025 different points, including the 100 vertices of P, that are a vertex of at least one triangle in T, and
- no three of these 2025 points are collinear.

A *quokkalateral* is a convex quadrilateral that consists of two triangles in T sharing a common side. Two quokkalaterals may share a triangle in T.

Determine the minimum number of quokkalaterals among all Aussie triangulations.

(China)

Answer: The minimum number of quokkalaterals is 1011.

Solution. We will solve the problem in general where P is a regular m-gon and there are exactly n different points that are a vertex of at least one triangle in T. We let q be the total number of quokkalaterals in such a configuration. So the problem is to find the minimum value of q, given that $n = 2025$ and $m = 100$.

Let the $n - m$ vertices which are not vertices of P be called *interior vertices*, and let the edges which are not edges of P be called *interior edges*. Consider the sum of the interior angles of all the triangles in T. Each of the interior vertices of the triangulation contributes 360° to the sum, and the vertices of P together contribute the sum of the interior angles of P, so

$$|T| \cdot 180^\circ = (n-m) \cdot 360^\circ + (m-2) \cdot 180^\circ \implies |T| = 2n - m - 2.$$

Let e be the number of interior edges, and count the three edges of each triangle in T. Each side of the m-gon is counted once, and each interior edge is counted twice, so

$$3|T| = m + 2e \implies e = 3n - 2m - 3.$$

We mark some of the interior edges with arrows as follows. Let xy be an interior edge, where xy belongs to the triangle xyz on one side and xyw on the other side.

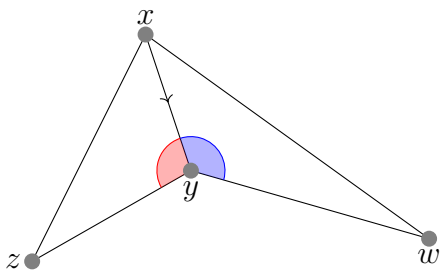


Figure 2: The direction of an arrow edge.

- Suppose that the quadrilateral $xzyw$ has an interior angle larger than 180° (it cannot have more than one). Since its interior angles at z and w are interior angles of triangles in T, they are both less than 180°, so exactly one of its interior angles at x and y exceeds 180°. Suppose that is the interior angle at y. We put an arrow on edge xy pointing to y, as shown in Figure 2, and we say y is the *head* of arrow $\overrightarrow{xy}$.
- If $xzyw$ is convex, then we do not mark the edge xy at all.

An arrow $\overrightarrow{xy}$ indicates that the two adjacent interior angles around y separated by the edge xy have a sum larger than $180°$. Now we make some trivial observations about the number of arrows that can point to a single vertex.

1. Because P is convex, only internal vertices can be the head of an arrow.
2. If a vertex y is the head of two different arrows $\overrightarrow{x_1y}$ and $\overrightarrow{x_2y}$, then x_1y and x_2y are two adjacent edges around y. Otherwise, the pair of interior angles around y separated by x_1y and the pair separated by x_2y are four disjoint angles around y with a sum larger than $360°$, which is impossible.
3. Consequently, any vertex y is the head of at most three different arrows, and if it is the head of three different arrows $\overrightarrow{x_1y}$, $\overrightarrow{x_2y}$ and $\overrightarrow{x_3y}$, then vertex y is not incident to any edge other than these three.

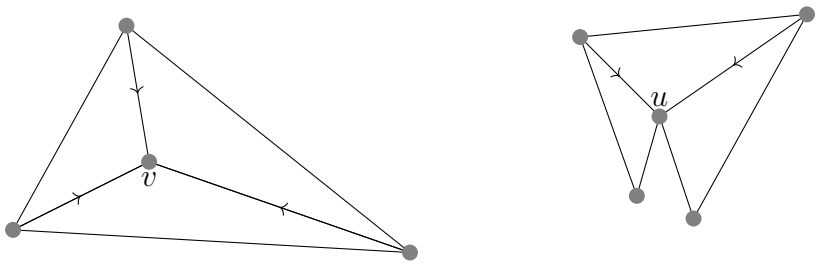


Figure 3: Points $v \in V_3$ and $u \in V_2$.

Let V_k denote the set of all internal vertices which are the head of exactly k arrows, for $k = 0, 1, 2, 3$. Examples are shown in Figure 3. Since the number of internal vertices is $n - m$ and each vertex is the head of at most 3 arrows this means

$$|V_3| + |V_2| + |V_1| + |V_0| = n - m.$$

Multiplying by 3 and ignoring some terms on the left hand side yields

$$3|V_3| + 3|V_2| + 2|V_1| \leqslant 3(n - m). \tag{1}$$

The problem asks us to minimise the number of internal edges with no arrow. Since the total number of edges is fixed ($e = 3n - 2m - 3$), this is equivalent to maximising the number of arrows. Counting the heads of arrows around each vertex, the quantity we are trying to maximise is

$$3|V_3| + 2|V_2| + |V_1|.$$

Now define an *optimal face* to be a triangle in T such that two of its sides are arrows with a common head (see Figure 4). For each vertex $v \in V_3$ there are three optimal faces around v, and for each vertex $u \in V_2$ there is one optimal face around u. So the total number of optimal faces is given by $3|V_3| + |V_2|$. Each optimal face is a triangle in T, so

$$3|V_3| + |V_2| \leqslant |T| = 2n - m - 2. \tag{2}$$

Summing equations (1) and (2) yields the upper bound we are looking for:

$$\begin{gathered}(3|V_3| + 3|V_2| + 2|V_1|) + (3|V_3| + |V_2|) \leqslant 3(n - m) + (2n - m - 2) \\ \implies \quad 3|V_3| + 2|V_2| + |V_1| \leqslant \frac{5n - 4m - 2}{2}.\end{gathered}$$

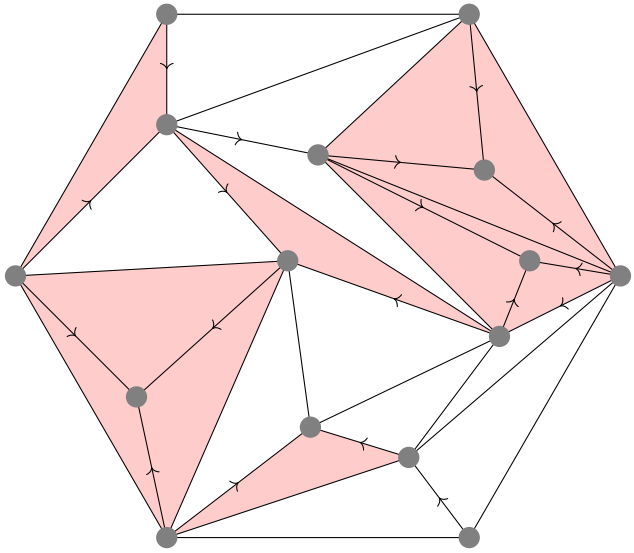


Figure 4: An Aussie triangulation, with optimal faces shaded.

The total number of internal edges is $e = 3n-2m-3$. Hence the total number of quokkalaterals (which is in one-to-one correspondence with the internal edges which are not arrows) is

$$q = e - (3|V_3| + 2|V_2| + |V_1|) \geqslant (3n - 2m - 3) - \frac{5n - 4m - 2}{2} = \frac{n-4}{2}.$$

Somewhat unexpectedly, this bound is independent of m. In the original question we had $n = 2025$ so the number of quokkalaterals in an Aussie triangulation is always at least 1011.

Finally, we provide an explicit Aussie triangulation with exactly 1011 quokkalaterals. For clarity, we give colours to the edges, and make sure the black edges are all arrows. First, construct diagonals that triangulate P with 97 blue diagonals and 98 triangles, using all the diagonals from a vertex A_1. Inside each $\triangle A_1A_iA_{i+1}$ $(i = 3, 4, \ldots, 99)$, add one vertex v_i and triangulate it with three black edges from v_i to A_1, A_i, and A_{i+1}, respectively. See the left hand diagram in Figure 5. We will not put extra vertices in $\triangle A_1A_iA_{i+1}$, so these black edges will be arrows. Note that we do not do this for $i = 2$, so the number of internal vertices so far is 97.

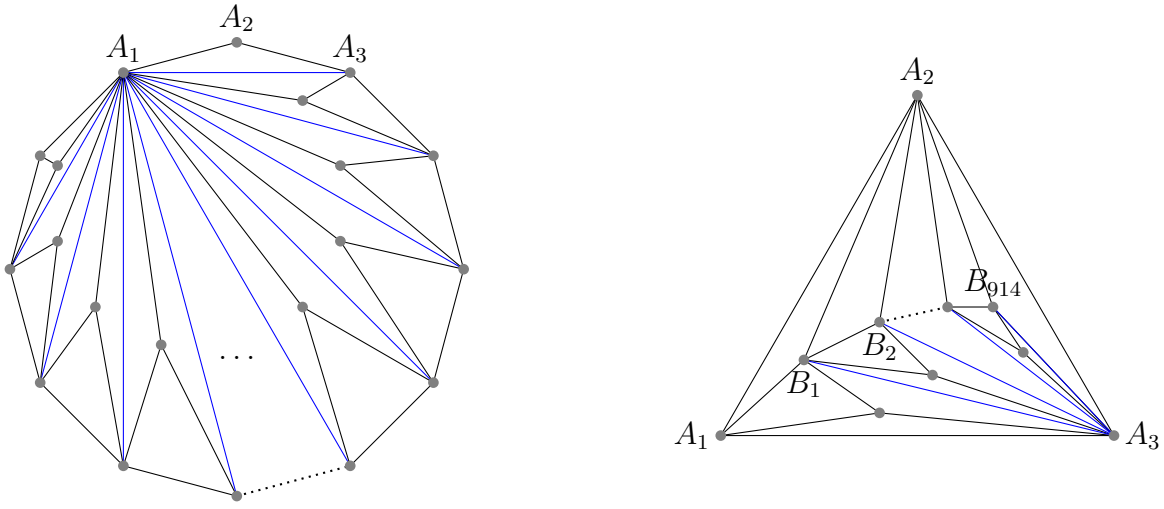


Figure 5: Construction of an Aussie triangulation with exactly 1011 quokkalaterals.

We still need to put $2025 - 100 - 97 = 2 \times 914$ vertices inside $\triangle A_1A_2A_3$. Since affine

transformations of the plane do not affect the convexity of any shape, for clarity we may draw $\triangle A_1A_2A_3$ as equilateral, as seen on the right in Figure 5.

Draw a convex arc from A_1 to A_3, and on the arc pick 914 points $B_1, B_2, \ldots, B_{914}$ in that order. Connect $A_1B_1, B_1B_2, \ldots, B_{913}B_{914}$ with black edges, connect A_2 to each B_i with a black edge, and connect A_3 to each B_i with a blue edge. Finally, add one point in each $\triangle B_iB_{i+1}A_3$, connected to B_i, B_{i+1} and A_3 by edges with the point as close to A_3 as necessary so that B_iB_{i+1} is an arrow directed towards B_{i+1}. Similarly add a point inside $\triangle A_1B_1A_3$. It is clear by the choice of these points that each black edge mentioned above is an arrow. The number of quokkalaterals is the number of blue edges, which is $97 + 914 = 1011$. Since we proved there cannot be fewer than 1011 quokkalaterals, this example completes the solution.

Comment. There are many other constructions achieving $q = \left\lceil \frac{n-4}{2} \right\rceil$ quokkalaterals. To find an optimal Aussie triangulation when $n \geqslant 2m - 2$, one simply tries to maximise the number of vertices in $V_2 \cup V_3$, and simultaneously have as many optimal triangles in T as possible.

- For even n, the bound $q = \left\lceil \frac{n-4}{2} \right\rceil$ is realised by any construction with $V_0 = V_1 = \varnothing$ in which every triangle in T is optimal.
- For odd n, the optimal constructions are of two types. Either $V_0 = V_1 = \varnothing$ and only one triangle in T is non-optimal; or $V_0 = \varnothing$, $|V_1| = 1$, and all triangles in T are optimal.

With a little extra analysis of the proof, particularly the inequalities (1) and (2), it can be seen that optimal Aussie triangulations must have $|V_2| = 914$ and $|V_3| = 1011$ (as in our example), or $|V_2| = 912$ and $|V_3| = 1012$, with one vertex in V_1.

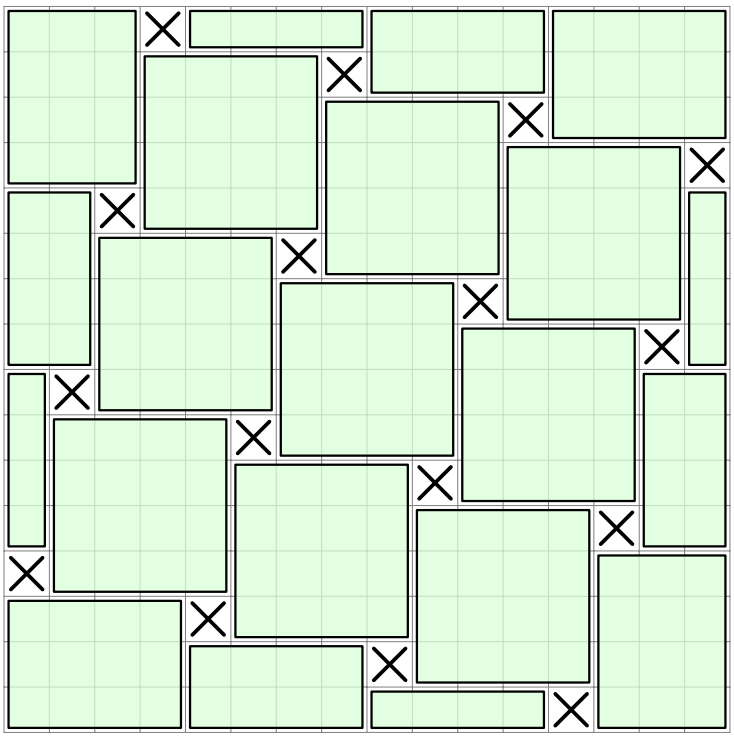
C8. Consider a 2025×2025 grid of unit squares. We place on the grid some number of rectangular tiles, possibly of different sizes, such that each side of every tile lies on a grid line and every unit square is covered by at most one tile.

Determine the minimum number of tiles we need to place so that each row and each column has exactly one unit square that is not covered by any tile.

(Singapore)

Answer: We need to place at least $2025 + 2 \times 45 - 3 = 2112$ rectangles.

Solution 1. We consider the more general case of an $n \times n$ grid where $n = k^2$ is a square number, and prove that the answer is $k^2 + 2k - 3$. For the construction, consider the following pattern, which we illustrate below in the case of $k = 4$. There are $(k-1)^2$ rectangles in the centre, and $k-1$ rectangles along each of the four sides, giving a total of $k^2 + 2k - 3$ rectangles.



For the lower bound, take any configuration of rectangles and label every exposed square with an X. Surround the original grid with four $n \times 1$ rectangles.

A *rising path* is a path travelling along the sides of the rectangles in the up and right directions. Note that we allow length zero rising paths. Let $x_1, x_2, \ldots, x_m$ be the longest possible sequence of Xs with the following properties:

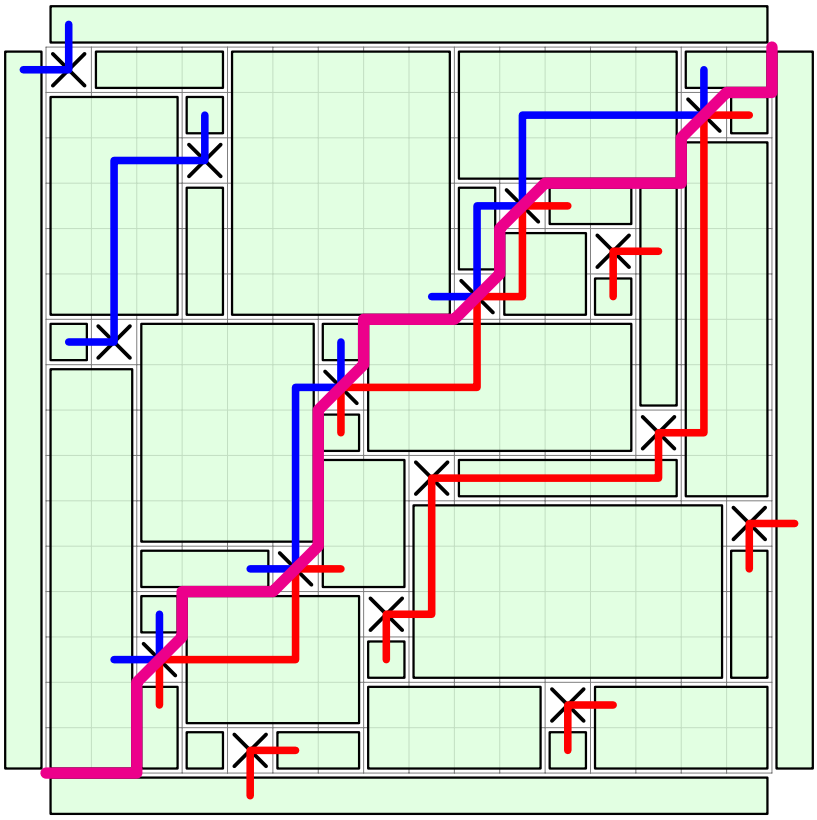
- There is a rising path connecting the bottom left corner of the grid with the bottom left corner of x_1.
- For each $i = 1, 2, \ldots, m-1$, there is a rising path connecting the top right corner of x_i with the bottom left corner of x_{i+1}.
- There is a rising path connecting the top right corner of x_m to the top right corner of the grid.

These rising paths and $x_1, x_2, \ldots, x_m$ together form a *great barrier reef* which divides the grid into two regions. Let $\mathcal{U}$ be the set of Xs in the upper region, let $\mathcal{L}$ be the set of Xs in the lower

region, and let $\mathcal{X}$ be the set $\{x_1, x_2, \ldots, x_m\}$ which are the Xs on the great barrier reef. Note that $\mathcal{U}$, $\mathcal{L}$ and $\mathcal{X}$ form a partition of the Xs.

For each X in $\mathcal{X} \cup \mathcal{L}$, we connect it to two rectangles — the one to the right of it and the one below it. This creates a number of disjoint *lower chains*. Since each X is connected to two rectangles and each rectangle is connected to at most two Xs, each chain must alternate between rectangles and Xs. Cycles are not possible, as the Xs form an increasing sequence, that is, each X is above and to the right of the previous X in the chain. So each chain must start and finish with a rectangle. Finally, all rectangles must be in the lower region, as they cannot cross the great barrier reef.

Similarly, for each X in $\mathcal{X} \cup \mathcal{U}$, we connect it to two rectangles — the one to the left of it and the one above it. This again creates a number of disjoint *upper chains*, where the rectangles are in the upper region. The properties of lower chains hold analogously for upper chains. The upper and lower chains must be disjoint, with the exception of the Xs in $\mathcal{X}$. The following diagram shows the great barrier reef as well as the upper and lower chains.



Claim. Each chain has at most m Xs.

Proof. Suppose there is a chain with Xs labelled $y_1, y_2, \ldots, y_l$ where $l > m$. We will show that they can be used in a greater barrier reef, which would violate the maximality of m.

By the construction of the chain, there is clearly a rising path between y_i and y_{i+1} for $i = 1, 2, \ldots, l-1$, via the rectangles in the chain. To find a connecting rising path between the bottom left corner of the grid and the bottom left corner of y_1, we note that in a rectangular tiling, it is always possible to either travel downward or leftward from a vertex. So we can start from the bottom left corner of y_1, moving either downward or leftward until we reach the bottom left corner of the grid. Similarly, the top right of y_m can always be connected by a rising path to the top right of the grid.

Thus we have constructed a greater barrier reef with more than m Xs, which is a contradiction. □

We return to the upper and lower chains. Every X in $\mathcal{U} \cup \mathcal{L}$ belongs to exactly one chain, while each of the m Xs in $\mathcal{X}$ belongs to exactly two chains. Counting the Xs in $\mathcal{X}$ twice, we have a total of $m + k^2$ Xs. By our claim, each chain can have at most m Xs, so we must have at least $(m + k^2)/m = 1 + k^2/m$ chains. Since each chain has exactly one more rectangle than Xs, and different chains have different rectangles, there must be at least

$$(m + k^2) + \left(1 + \frac{k^2}{m}\right) = k^2 + \left(m + \frac{k^2}{m}\right) + 1 \geqslant k^2 + 2k + 1$$

rectangles. Removing the four rectangles we added initially on the outside, there must be at least $k^2 + 2k - 3$ rectangles, as required.

Comment. This problem admits a much nicer answer and construction than it does in the general case, because of the fact that 2025 is perfect square. If 2025 is replaced by a general n, the solution above can be adapted to show that the bound is given by $\lceil n + 2\sqrt{n} - 3 \rceil$, and the construction is more complicated.

Solution 2. We provide a different proof of the lower bound. Once again, let $n = k^2$, and take a configuration of rectangles and label every exposed square with an X. Add two additional $n \times 1$ rectangles beyond the left and right sides of the grid, completely covering those sides. We will draw blue and red *waterfalls* which flow along the sides of the rectangles and through the Xs. The red waterfalls will flow in the down and right directions, while the blue waterfalls will flow in the down and left directions. We draw them according the following rules:

1. From each X, a red waterfall flows to the rectangle to its right, and then flows downward to the bottom left corner of this rectangle. Similarly, from each X, a blue waterfall flows from it to the rectangle to its left, and then flows downward to the bottom right corner of this rectangle.

2. Consider each rectangle R that has an X directly below. If there is a red waterfall in the bottom left corner of R, extend the waterfall rightward until it flows into the X. Similarly, if there is a blue waterfall in the bottom right corner of R, extend it leftward until it flows into the X.

3. Consider each rectangle R that does not have an X directly below, and has both a red waterfall in the bottom left corner and a blue waterfall in the bottom right corner.

   (a) Suppose that the X on the bottom row of R (that is, standing on the same horizontal grid line) is to the left of R. Check the bottom right corner of R. If there is a rectangle S below R such that their right sides are aligned, then extend the blue waterfall downward until it reaches the bottom right corner of S. Otherwise, there must be a rectangle T to the right of R such that their bottom sides are aligned. In this case, extend the red waterfall rightward until it reaches the bottom left corner of T. See Figure 6 for the two possible cases.

   (b) In the case that the X on the bottom row of R is to the right of R, we perform analogous waterfall extensions, but swapping left with right, and red with blue.

These steps are carried out repeatedly until no more waterfalls can be drawn or extended. From the rules, we see that each red waterfall flows downward from an X, makes at most one turn rightward, before possibly flowing into another X. Each blue waterfall flows downward from an X, makes at most one turn leftward, before possibly flowing into another X.

Claim 2.1. A blue and a red waterfall cannot flow along the same line segment.

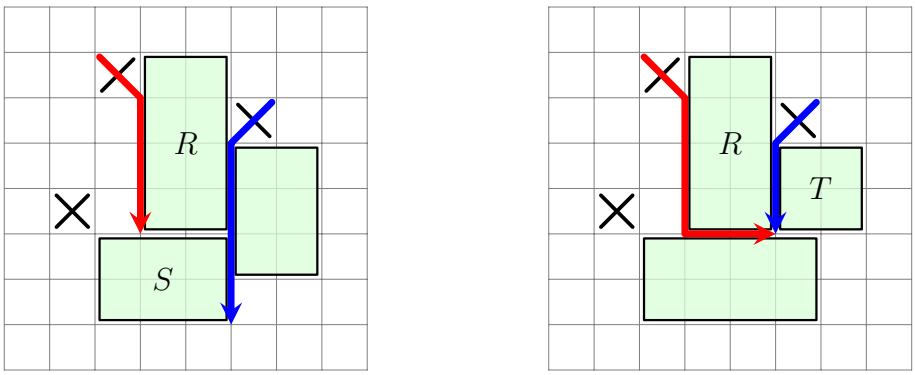


Figure 6: The two possible applications of Rule 3(a), depending on the tiling at the bottom right corner of R. Only one of the waterfalls is extended.

Proof. Suppose a blue and a red waterfall flow along the same line segment. If the segment is vertical, then the waterfalls must have travelled from the two X s on the two sides of this vertical line. However, by Rule 2, the waterfall from the higher X must turn before passing the lower X, so this is impossible.

If the two waterfalls flow along the same horizontal segment, they do so along the bottom side of a rectangle. But this cannot occur during an application of Rule 2, because they will flow into the X below the rectangle before meeting; and nor can it occur during an application of Rule 3, as only one of the two waterfalls will flow horizontally. So this is also impossible. □

Claim 2.2. Two waterfalls of the same colour cannot intersect.

Proof. Two waterfalls cannot merge while they are still travelling vertically, as they would have to have travelled from two X s in the same column.

Since each waterfall makes at most one turn (from vertical to horizontal) before flowing into the next X or terminating, it remains to check the possibility of a vertically flowing waterfall merging with a horizontally flowing waterfall. Without loss of generality, suppose two blue waterfalls merge at the bottom left corner of rectangle R. The horizontally moving blue waterfall can only reach the bottom left corner of R via Rule 3, which implies that there must be a red waterfall flowing downward on the left side of R. By Claim 2.1 there cannot also be a blue waterfall flowing downward on the left side of R, so this is impossible. □

Each red waterfall must terminate at the bottom left corner of a rectangle, and each blue waterfall must terminate at the bottom right corner of a rectangle. Let such rectangles be the *basins* of the respective waterfalls. Figure 7 shows the blue and red waterfalls as well as their basins.

Claim 2.3. No two waterfalls have the same basin.

Proof. By Claim 2.2, two waterfalls of the same colour cannot intersect, so they cannot have the same basin. If two waterfalls of different colours have the same basin, then the red waterfall must end at the bottom left corner and the blue waterfall must end at the bottom right corner of the common basin. But this contradicts Rule 3, which says that one of the waterfalls will be extended. □

We return to the problem at hand. Every X belongs to one red waterfall and one blue waterfall. Since blue waterfalls flow down and left, while red waterfalls flow down and right, a blue waterfall and a red waterfall cannot share more than one X. Suppose there are b blue waterfalls and r red waterfalls. Then there must be a blue waterfall flowing through at least k^2/b X s. Since these X s must belong to distinct red waterfalls, we must have $r \geqslant k^2/b$. Hence

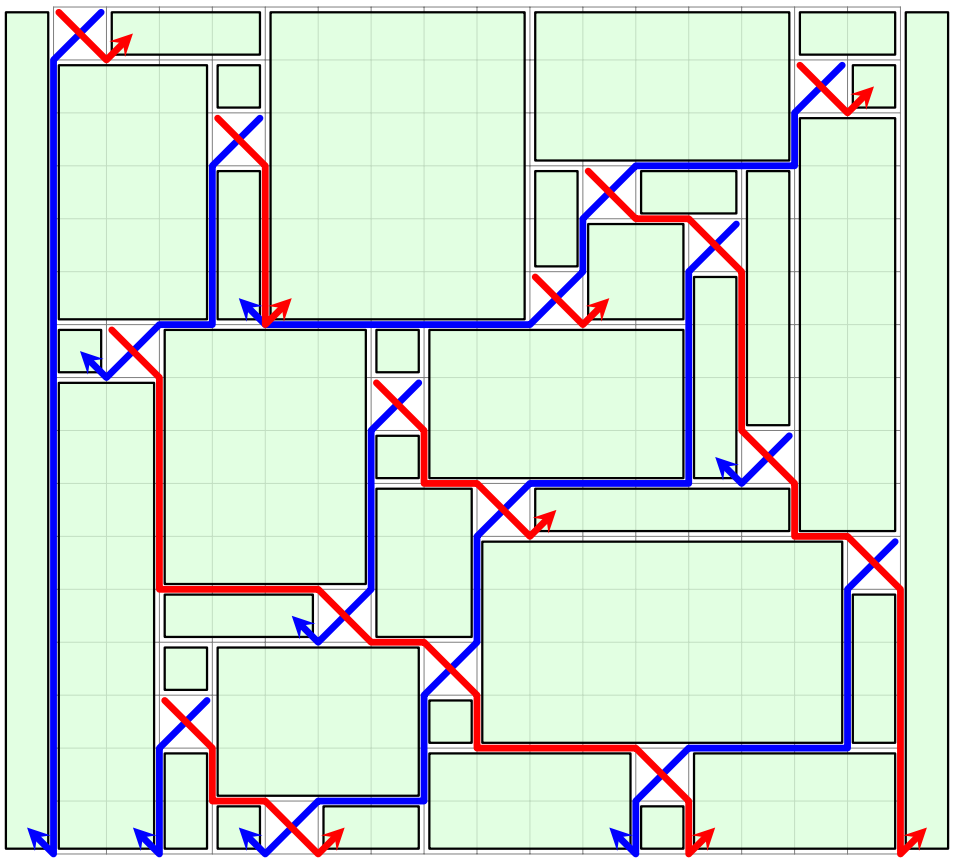


Figure 7: The blue and red waterfalls are shown above. The arrow at the end of each waterfall points toward its basin.

by Claim 2.3, in total, there must be at least

$$b + r \geqslant b + \frac{k^2}{b} \geqslant 2k$$

basins.

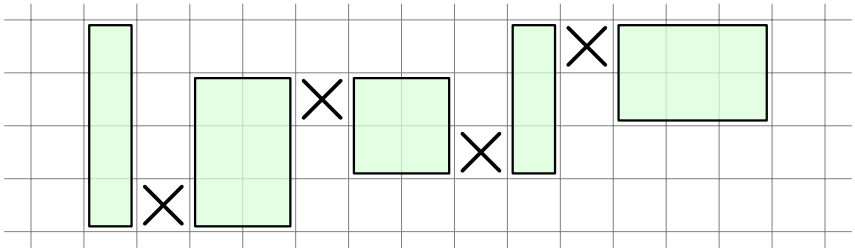
There are $k^2 - 1$ rectangles that have an X directly below. By Rule 2, a basin cannot have an X directly below, giving us at least $2k$ additional rectangles. Removing the two rectangles outside of the original grid, we must have at least $k^2 - 1 + 2k - 2 = k^2 + 2k - 3$ rectangles in total, completing the solution.

Solution 3. We provide another proof of the lower bound. As in the first solution, we write $k = 45$ so that $n = k^2$, and surround the $n \times n$ grid with four $n \times 1$ rectangles.

We note that every exposed square marked X has a unique rectangle adjacent to the left and a unique rectangle adjacent to the right. Because no two Xs are in the same column, every rectangle has at most one X adjacent to the left, and at most one X adjacent to the right. We define a *buoy line* to be a sequence

$$R_1, X_1, R_2, X_2, R_3, \ldots, R_m, X_m, R_{m+1},$$

where R_i is the rectangle adjacent to the X-marked square X_i, on the left of X_i; R_{i+1} is the rectangle adjacent to X_i, on the right of X_i; R_1 does not have an X adjacent to the left; and R_{m+1} does not have an X adjacent to the right. We allow $m = 0$, in which case the buoy line is just a single rectangle R_1. Then every X and every rectangle is contained in a unique buoy line.



Note that every nonempty intersection of a column with a buoy line is a contiguous block of unit squares. We define a binary relation $\prec$ on the set of buoy lines, where we say that $C \prec C'$ when

- $C \neq C'$ and
- there exists a column I intersecting both C and C' for which $I \cap C'$ lies above $I \cap C$.

Claim 3.1. Let C and C' be two buoy lines satisfying $C \prec C'$. If a column I intersects both C and C', then $I \cap C'$ lies above $I \cap C$.

Proof. By definition, the statement is true for some column I. Because the set of columns intersecting both C and C' form a contiguous block of columns, it suffices to show that if I, I' are adjacent columns and the statement is true for I, then it is also true for I'. This is clear as $I \cap C$ must be connected to $I' \cap C$ and similarly for C'. □

Claim 3.2. The binary relation $\prec$ can be extended to a linear order on the set of buoy lines.

Proof. We prove by induction on n that for any set S of buoy lines of size n, the binary relation $\prec$ extends to a linear order on S. The case $n = 1$ is clear. We consider the case $n \geqslant 2$, assuming the statement holds for smaller values of n.

If there exists a column that intersects all buoy lines in S, then $\prec$ is already a total order by Claim 3.1 and hence we are done. Otherwise, there exists a vertical line L with the property that the subset $A \subseteq S$ of buoy lines completely on the left of L is nonempty, and also the subset $B \subseteq S$ of buoy lines completely on the right of L is nonempty. It is clear that $A \cap B = \varnothing$ and $\prec$ restricted to $S - (A \cup B)$ is a linear order. By the inductive hypothesis, the relation $\prec$ on $S - A$ can be extended to a total order, and the relation $\prec$ on $S - B$ can be extended to a total order. On the other hand, elements of A and B are incomparable to each other. This shows that we can extend $\prec$ on S into a total order by combining the total orders on $S - A$ and $S - B$. □

We now label the buoy lines as

$$C_0, C_1, \ldots, C_\alpha,$$

so that $C_i \prec C_j$ implies $i < j$. Let m_i be the number of Xs contained in C_i. Then C_i contains $m_i + 1$ rectangles, and $m_0 + \cdots + m_\alpha = n$. Hence the total number of rectangles is $n + \alpha + 1$.

Claim 3.3. For every $0 \leqslant i \leqslant \alpha$ we have $m_i \leqslant \min(i, \alpha - i)$.

Proof. For every X in $C_0, \ldots, C_i$, there exists a unique rectangle adjacent to it below. Such rectangles are all distinct, because none of the Xs are in the same row, and each of these rectangles is in one of $C_0, \ldots, C_{i-1}$. This shows that

$$m_0 + \cdots + m_i \leqslant (m_0 + 1) + \cdots + (m_{i-1} + 1),$$

and hence $m_i \leqslant i$. By a similar argument, but using adjacent rectangles above X-marked squares, we see that $m_i \leqslant \alpha - i$. □

We now observe that $\alpha \geqslant 2k = 2\sqrt{n}$, because otherwise

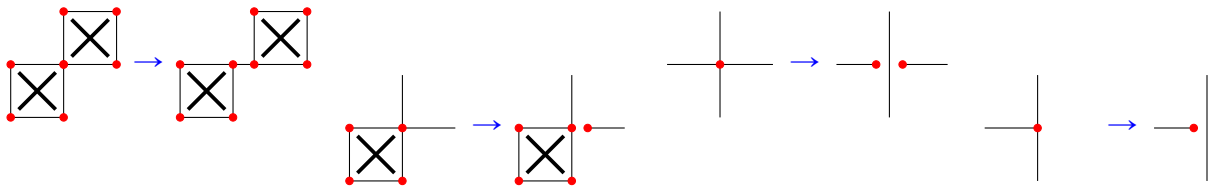
$$n = m_0 + \cdots + m_\alpha \leqslant (0 + 1 + \cdots + (k-1)) + ((k-1) + \cdots + 0) = k^2 - k$$

leads to a contradiction. We conclude that the number of rectangles $n + \alpha + 1$ is at least $k^2 + 2k + 1$, and hence after removing the four auxiliary $n \times 1$ rectangles, the number of rectangles is at least $k^2 + 2k - 3$.

Solution 4. As with the other solutions, let us take a configuration of rectangles and label every exposed square with an X. We treat it as a planar graph, and let L, V, E and F be the number of leaves, vertices, edges and faces respectively (recall that leaves are vertices incident with only one edge). Initially we observe that $L = 0$ in our graph. By Euler's formula $V - E + F = 1$, where we are excluding the outside face. Since F is the number of Xs and rectangles, it is sufficient to show that $F \geqslant 2n + 2\sqrt{n} - 3$, or equivalently,

$$E - V + L \geqslant 2n + 2\sqrt{n} - 4.$$

We now perform a series of alterations to the graph as shown below, in which the value of $E - V + L$ is invariant.



- For each pair of Xs that are diagonally adjacent (have a corner in common) replace the corner with a pair of vertices joined by a horizontal edge.

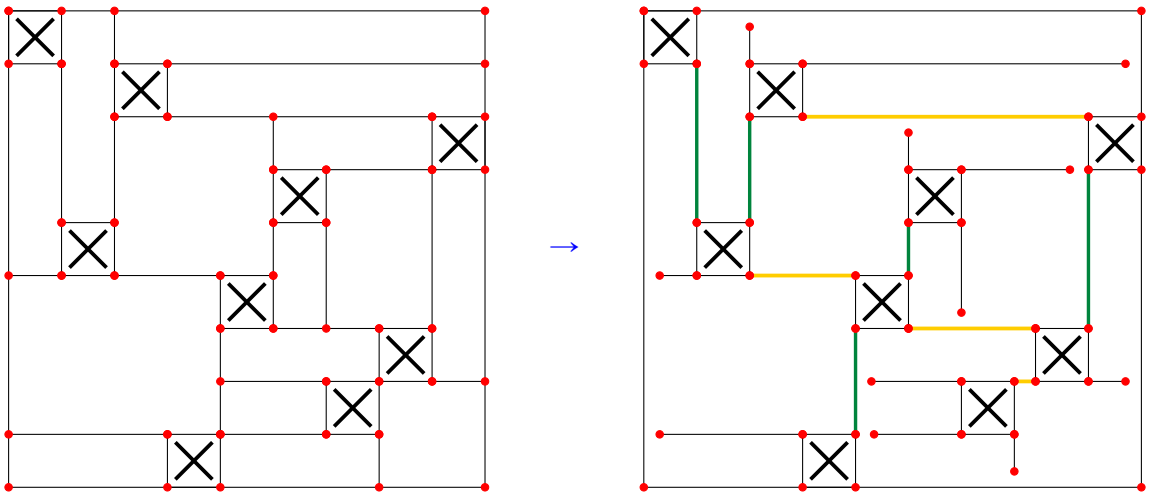
- For each corner of an X that is a vertex of degree 4 but not the corner of any other X, retract the horizontal edge connecting to the X at this corner to form a new vertex of degree 1 and reduce the degree of the corner from 4 to 3.

- For each vertex of degree 4 that is not the vertex of an X, retract the two incident horizontal edges, forming two new vertices, and merge the two vertical edges into a single edge, removing the original vertex.

- For each vertex of degree 3 that is not the vertex of an X, retract the edge with no opposing edge, forming one new vertex, and merge the other two edges, removing the original vertex.

One can verify that the value of $E - V + L$ is invariant under each such alteration. After performing all possible alterations we arrive at a new planar graph. Call this new graph G.

We now colour the edges in G which join the corners of two different Xs. Colour these edges *green* if they are vertical, and colour them *gold* if they are horizontal. The alterations we performed earlier ensure that no green edge intersects with a gold edge. Observe that the green edges link the Xs into *green chains*. Note that a green chain may consist of a single X incident with no green edges. Define *gold chains* analogously.



The original graph

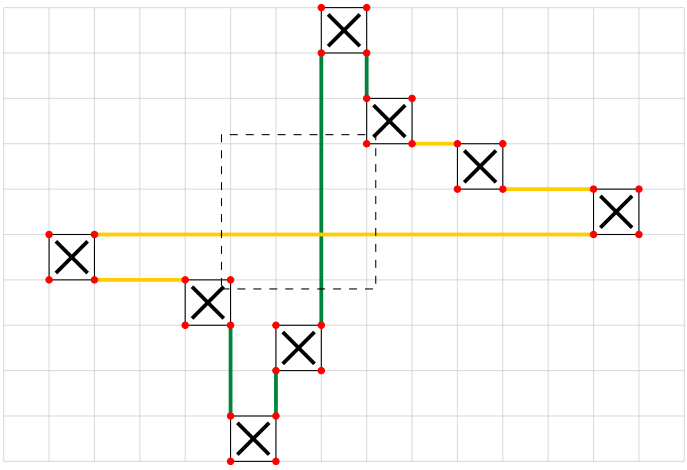
The graph G

Lemma. A gold chain and a green chain can intersect in at most one X.

Proof. Let $(1, \sigma(1)), \ldots, (n, \sigma(n))$ be the coordinates of the Xs. For the sake of contradiction, assume we have green and gold chains intersecting at two different Xs. Pick a minimal example of this — a pair of Xs $(i, \sigma(i))$ and $(j, \sigma(j))$ with $i < j$ in the same green and gold chains with minimal $j - i$. Without loss of generality, suppose that $\sigma(i) < \sigma(j)$. Note that the x-coordinates of the Xs in a green chain are consecutive numbers, while the y-coordinates of the Xs in a gold chain are consecutive numbers. Consider the green and gold chains linking these two Xs:

$$\text{green chain} : (i, \sigma(i)),\ (i+1, \sigma(i+1)),\ \ldots,\ (j, \sigma(j))$$
$$\text{gold chain} : (i, \sigma(i)),\ (\sigma^{-1}(\sigma(i)+1), \sigma(i)+1),\ \ldots,\ (j, \sigma(j))$$

By the minimality of $j-i$, there cannot be an X at $(l, \sigma(l))$ with $i < l < j$ and $\sigma(i) < \sigma(l) < \sigma(j)$, because otherwise it would appear in both chains. Hence, there cannot be any Xs within the rectangle $[i+1, j-1] \times [\sigma(i)+1, \sigma(j)-1]$, shown dotted in the diagram below.



However, at some point, the gold chain must cross the dotted rectangle horizontally and the green chain must cross the dotted rectangle vertically (possibly along the boundary of the dotted rectangle), which contradicts the fact that no green and gold edge can touch by our construction. □

We return to the original problem. Let a be the number of green chains and let b be the number of gold chains. Each X lies on exactly one green and exactly one gold chain, so by the lemma we get $n \leqslant ab$. Note that a green chain with t Xs consists of $(t-1)$ green edges, and since the total number of Xs is n, this means there are $(n-a)$ green edges in total. Similarly

there are $(n-b)$ gold edges in total. Let $g = (n-a) + (n-b)$ be the total number of green and gold edges. By the AM-GM inequality we get

$$g = 2n - (a+b) \leqslant 2n - 2\sqrt{ab} \leqslant 2n - 2\sqrt{n}. \tag{1}$$

Suppose c of the X s are in the corners of the grid. Each vertex of G is either a vertex of an X, a corner of the grid, or a leaf. Hence,

$$V = 4n + 4 - c + L.$$

Next, we bound E below by only counting the edges incident to an X. Each X has four sides plus an additional edge incident to each corner, unless this corner is the corner of the grid. The edges which are incident to two different X s are precisely the green and gold edges. Hence,

$$E \geqslant 4n + 4n - c - g = 8n - g - c.$$

Putting this all together yields

$$E - V + L \geqslant \big(8n - g - c\big) - \big(4n + 4 - c + L\big) + L = 4n - g - 4.$$

Finally, using equation (1) we get

$$E - V + L \geqslant 4n - g - 4 \geqslant 4n - (2n - 2\sqrt{n}) - 4 = 2n + 2\sqrt{n} - 4$$

as required.

Comment. Problem 5 of Romanian Master of Mathematics 2017 asked a seemingly similar problem, where the rectangular tiles are replaced by $1 \times m$ tiles. However, that was a vastly simpler and easier problem and the ideas used are not at all helpful for solving this problem.

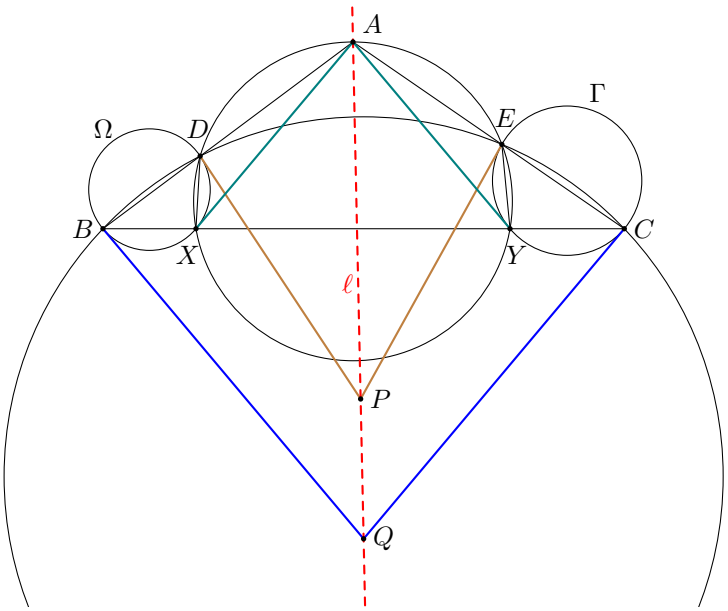
Geometry

G1. Let ABC be a triangle such that $\angle A$ is obtuse. Let D and E be points on sides AB and AC, respectively, such that $BDEC$ is cyclic. Suppose the circumcircle of triangle ADE intersects side BC at two points X and Y, where B, X, Y and C lie in that order. Let the circumcircles of triangles BDX and CEY be Ω and Γ, respectively. Let P be a point such that PD is tangent to Ω and PE is tangent to Γ. Let Q be a point such that QB is tangent to Ω and QC is tangent to Γ.

Prove that points A, P and Q are collinear.

(Czech Republic)

Solution 1. Define ℓ to be the radical axis of Ω and Γ. Note that a point lies on ℓ if its tangents to Ω and Γ are equal in length. First, we will show that A lies on ℓ. Second, we will show that $AX = AY$. Third, we will show that Q lies on ℓ. Fourth, we will show that P lies on ℓ. Together, these will imply that A, P, and Q are collinear.



First, we show that A lies on the radical axis ℓ. This is because $BDEC$ is cyclic, so $AB \cdot AD = AC \cdot AE$, which means the power of A with respect to Γ and Ω are equal.

Second, we show that $AX = AY$. Since $AEXD$ and $BDEC$ are cyclic, we have $\angle AXD = \angle AED = \angle CBD$. Therefore, AX is tangent to Ω. Similarly, AY is tangent to Γ. Since A lies on ℓ, we have $AX = AY$.

Third, we show that Q lies on the radical axis ℓ. It suffices to show that $QB = QC$, or $\angle QBC = \angle BCQ$. This follows from

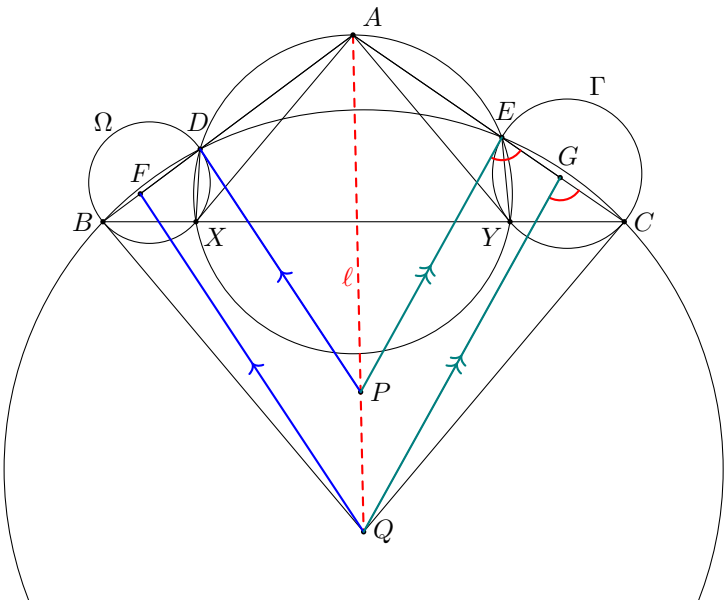
$$\angle QBC = \angle BDX = \angle AYX = \angle YXA = \angle YEC = \angle BCQ.$$

Fourth, we show that P lies on the radical axis ℓ. Note that $\angle PDA = \angle DXB$, since PD is tangent to Ω. To prove P lies on ℓ, it suffices to show $PD = PE$, or $\angle PDE = \angle DEP$. This follows from

$$\angle PDE = \angle PDA - \angle EDA = \angle DXB - \angle ECB = \angle DEY - \angle PEY = \angle DEP.$$

Solution 2. We provide an alternative solution that does not rely on the properties of the radical axis. We begin by proving $QB = QC$ and $PD = PE$ in the same way as Solution 1.

Define F on line AB such that $DP \parallel FQ$. Then $\angle BFQ = \angle BDP = \angle QBD = \angle QBF$, where the second equality follows from QB and PD being tangent to Ω. Therefore, $QB = QF$. Define G on line AC such that $EP \parallel GQ$. It follows similarly that $QC = QG$.



Consider triangles PDE and QFG. We have $PD = PE$, and $QF = QB = QC = QG$. Moreover, $PD \parallel QF$ and $PE \parallel QG$. Therefore, PDE and QFG are similar isosceles triangles. There is a dilation centred at A sending $\triangle PDE$ to $\triangle QFG$, so A, P and Q are collinear.

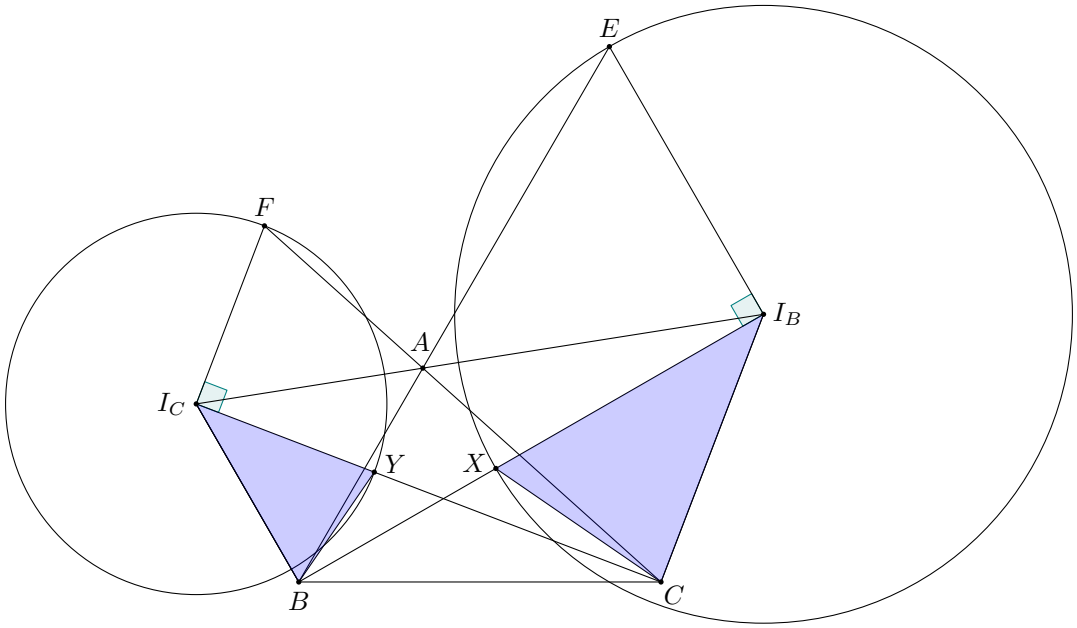
G2. Let ABC be an acute-angled triangle. Let I_B and I_C be the excentres of triangle ABC opposite B and C, respectively. Let E be a point on line BA such that $\angle EI_BB = 90^\circ$. Let F be a point on line CA such that $\angle CI_CF = 90^\circ$. Let the circle with centre I_B and radius I_BE intersect the segment I_BB at X. Let the circle with centre I_C and radius I_CF intersect the segment I_CC at Y.

Prove that BY is perpendicular to CX.

The excentre *of a triangle ABC opposite vertex B is the centre of the circle that is tangent to line segment AC, to ray BA beyond A, and to ray BC beyond C. The excentre opposite C is similarly defined.*

(Islamic Republic of Iran)

Solution 1. We will prove that $\triangle BI_CY \sim \triangle XI_BC$. It will follow that $BY \perp CX$, since we know $I_CB \perp XI_B$ and $I_CY \perp CI_B$.

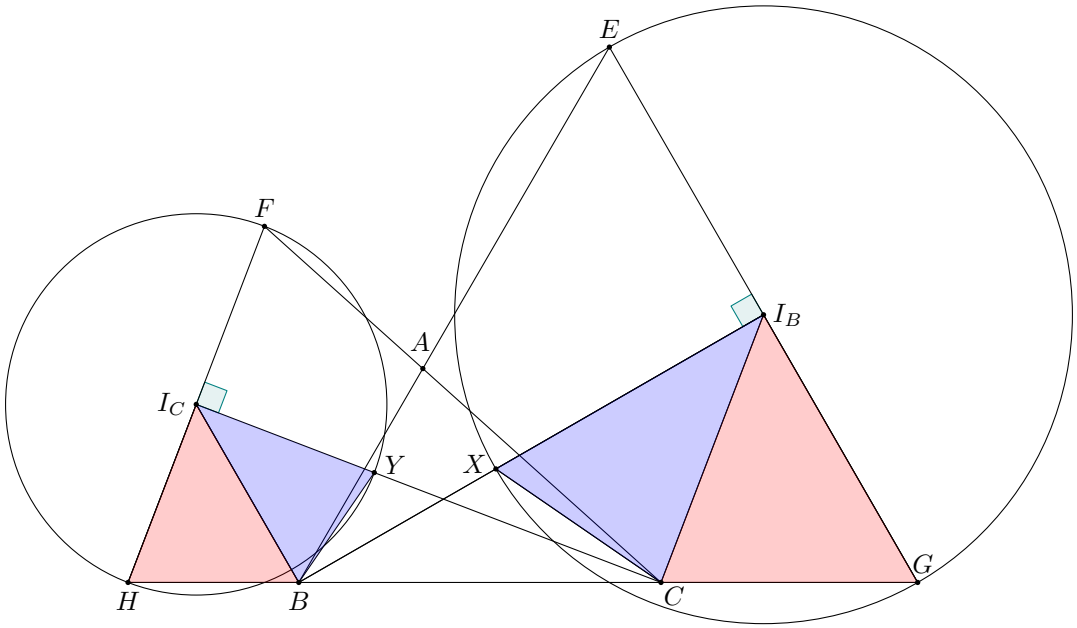


First, we will prove that $\angle BI_CY = \angle XI_BC$. This follows from the fact that that $\angle I_BBI_C = \angle I_BCI_C = 90^\circ$, so I_CI_BCB is cyclic.

Second, we will prove that $\dfrac{BI_C}{XI_B} = \dfrac{I_CY}{I_BC}$, which with the above, proves $\triangle BI_CY \sim \triangle XI_BC$. Note that I_B, A, I_C are collinear. Moreover, we have $I_CB \parallel EI_B$, since BI_B is perpendicular to both of them. This means that $\triangle BI_CA \sim \triangle EI_BA$, and we similarly have $\triangle I_CFA \sim \triangle I_BCA$. Therefore,

$$\frac{BI_C}{XI_B} = \frac{BI_C}{EI_B} = \frac{I_CA}{I_BA} = \frac{I_CF}{I_BC} = \frac{I_CY}{I_BC}.$$

Solution 2. We provide an alternative way to obtain $\dfrac{BI_C}{XI_B} = \dfrac{I_CY}{I_BC}$.
Let EI_B intersect BC at G. Let FI_C intersect BC at H.

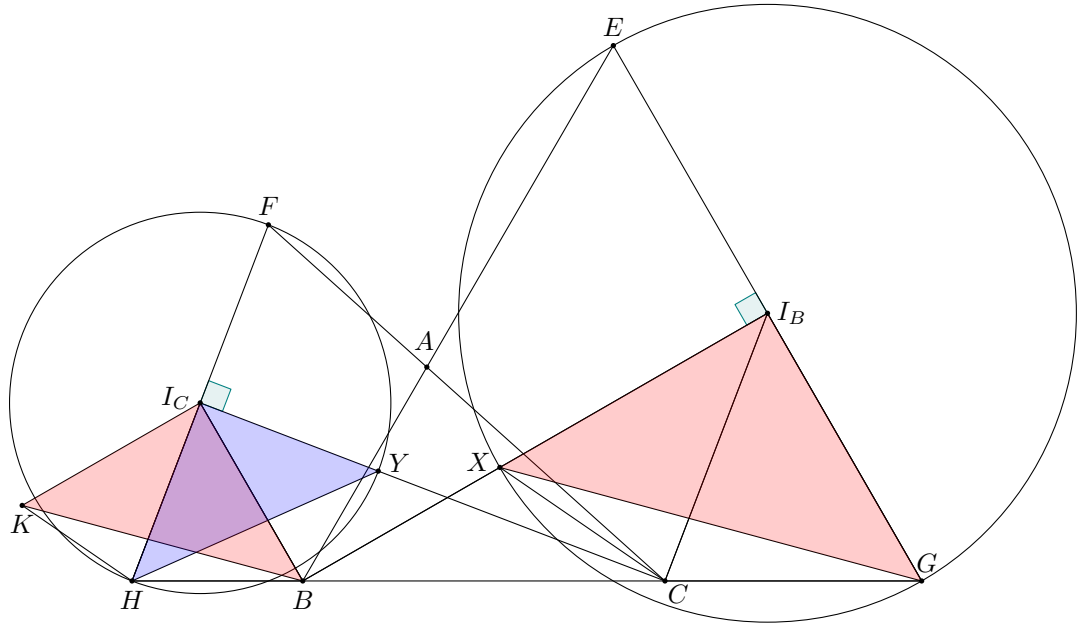


First, we will prove that $\triangle BHI_C \sim \triangle GCI_B$. Since $GI_B \perp BI_B$ and $BI_B \perp BI_C$, we have $GI_B \parallel BI_C$. Similarly, $CI_B \parallel HI_C$. Also, $BH \parallel GC$. Therefore, $\triangle BHI_C \sim \triangle GCI_B$.

Since $\angle FCI_C = \angle I_CCH$, $\angle CI_CF = \angle HI_CC$, and $I_CC = I_CC$, we get $\triangle HI_CC \cong \triangle FI_CC$. Therefore, $HI_C = FI_C = YI_C$. Similarly, $GI_B = XI_B$. Combining with $\triangle BHI_C \sim \triangle GCI_B$:

$$\frac{BI_C}{I_CY} = \frac{BI_C}{I_CH} = \frac{GI_B}{I_BC} = \frac{XI_B}{I_BC}.$$

Solution 3. We begin similarly to Solution 2. Let EI_B intersect BC at G. Let FI_C intersect BC at H. Then $HI_C = YI_C$, $GI_B = XI_B$, which implies that $\triangle I_CYH$ and $\triangle I_BGX$ are right-angled isoceles triangles. Moreover, from Solution 2, we have $\triangle BHI_C \sim \triangle GCI_B$.



Construct K so that I_CBK is a right-angled isosceles triangle. Then $I_CBHK \sim I_BGCX$, which implies $HK \parallel CX$.

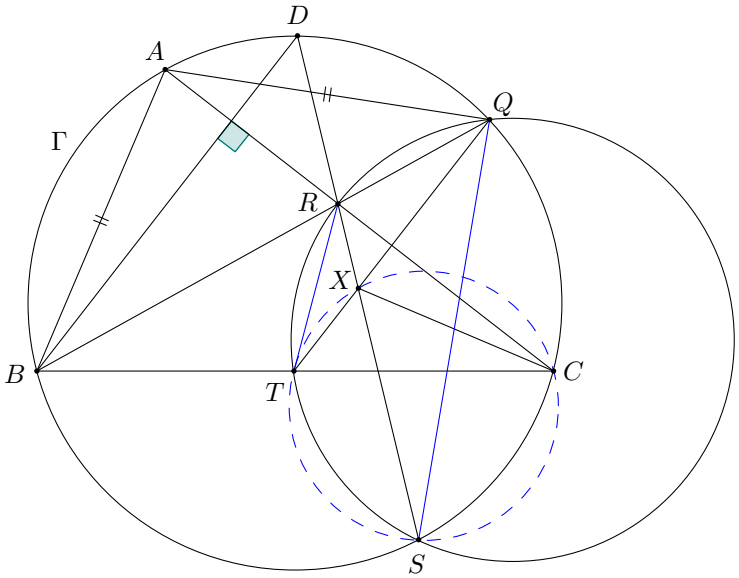
Finally, in $\triangle I_CHK$ and $\triangle I_CYB$, the pairs of sides (I_CK, I_CB) and (I_CH, I_CY) are equal and perpendicular. Therefore, the final pair of sides (KH, BY) are equal and perpendicular. So $HK \perp BY$. Combining this with $HK \parallel CX$ yields $BY \perp CX$.

G3. Let ABC be an acute-angled triangle with $AB < AC$. Let Γ be the circumcircle of triangle ABC. The perpendicular from B to AC intersects Γ again at $D \neq B$. Let Q be the point on Γ such that $Q \neq B$ and $AB = AQ$. Lines BQ and AC intersect at R. Line DR intersects Γ again at $S \neq D$. Segment BC intersects the circumcircle of triangle QRS at T. Lines TQ and RS intersect at X.

Prove that CX is perpendicular to AB.

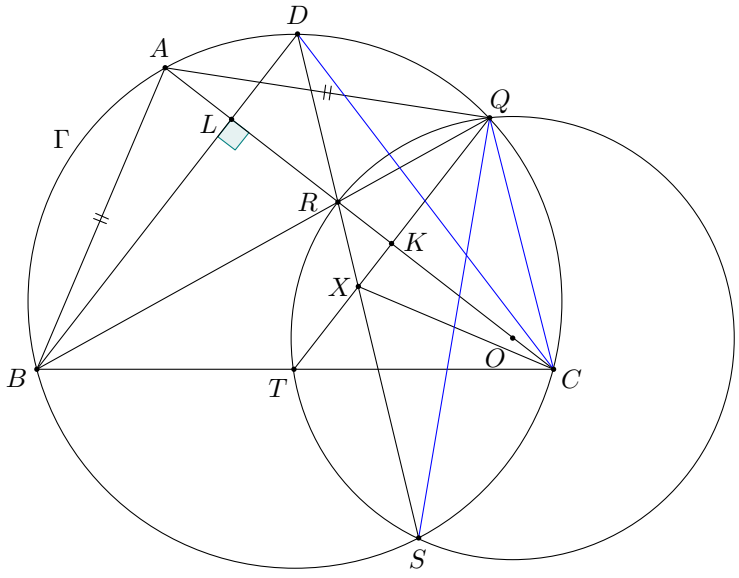
(Kosovo)

Solution 1. Let T' be the reflection of Q across CR. Then $RQ = RT'$ and $BD \parallel T'Q$. Therefore, $\angle QSR = \angle QBD = \angle RQT' = \angle QT'R$, so $RQST'$ is cyclic. Since AB and AQ subtend equal angles at C, point T' lies on BC. It follows that $T' = T$.



As $BD \parallel TQ$ and $DCSB$ is cyclic, we have $\angle CTX = \angle CBD = \angle CSX$, so $XCST$ is cyclic. Thus $\angle XCT = \angle XST$. As $RQST$ is cyclic, we have $\angle XST = \angle RQT$. As BQT is on a circle with centre A, we have $\angle RQT = \frac{1}{2}\angle BAT = 90^\circ - \angle TBA$. Hence $\angle XCT = 90^\circ - \angle CBA$, which implies that $CX \perp AB$, as required.

Solution 2. Let O be the circumcentre of triangle QSR. Then $\angle ORQ = 90^\circ - \frac{1}{2}\angle QOR = 90^\circ - \angle QSR = 90^\circ - \angle QBD = \angle ARB$, so line AC passes through O. Since $AB = AQ$, the line AC bisects $\angle QCT$. It now follows that $AC \perp TQ$.



From here, we have $TQ \parallel BD$. Let L and K be the points of intersection of BD and QT, respectively, with line AC. This gives rise to the pairs of similar triangles $\triangle RKX \sim \triangle RLD$, $\triangle RQK \sim \triangle RBL$ and $\triangle CBL \sim \triangle CQK$. Thus

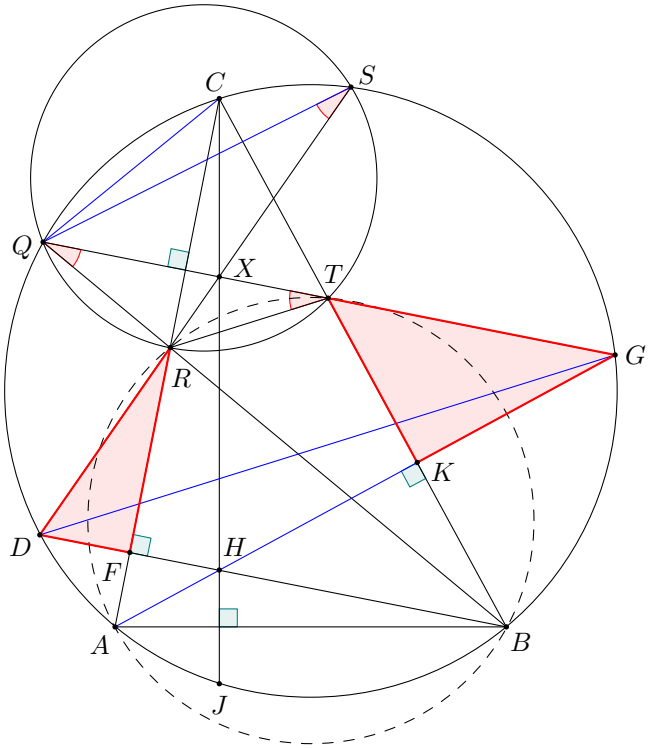
$$\frac{KX}{LD} = \frac{KR}{LR} = \frac{KQ}{LB} = \frac{CK}{CL},$$

so $\triangle CXK \sim \triangle CDL$, which gives $\angle ACX = \angle DCA = \angle DBA$. As $AC \perp BD$, it now follows that $CX \perp AB$, as required.

Solution 3. Let H be the orthocentre of triangle ABC. Then D is the reflection of H across the side AC. Let G and J be the reflections of H across the sides BC and AB, respectively. Then $AD = AJ$ and $BJ = BG$. Since $AB = AQ$, it follows that $DQ = BJ = BG$. Thus $QG \parallel BD \perp AC$.

Let T' be the second point of intersection between circle QRS and line QG. Then $\angle QT'R = \angle QSR = \angle QSD$. Since $DQ = BG$, we have $\angle QSD = \angle BQG = \angle RQT'$. Thus $\angle QT'R = \angle RQT'$, and so $RQ = RT'$. Then $\angle ACT' = \angle RCT' = \angle QCR = \angle QCA = \angle ACB$. Thus point T' is on line BC, and therefore $T' = T$.

Consider the triangles DFR and GKT. Since points F and K are midpoints of HD and HG, respectively, we have $FK \parallel DG$. Since $\angle QTR = \angle QSD = \angle QGD$, we have $RT \parallel DG$. Thus the triangles DFR and GKT are in perspective and so, by Desargues' Theorem, the point X lies on the line CH. Hence $CX \perp AB$, as required.



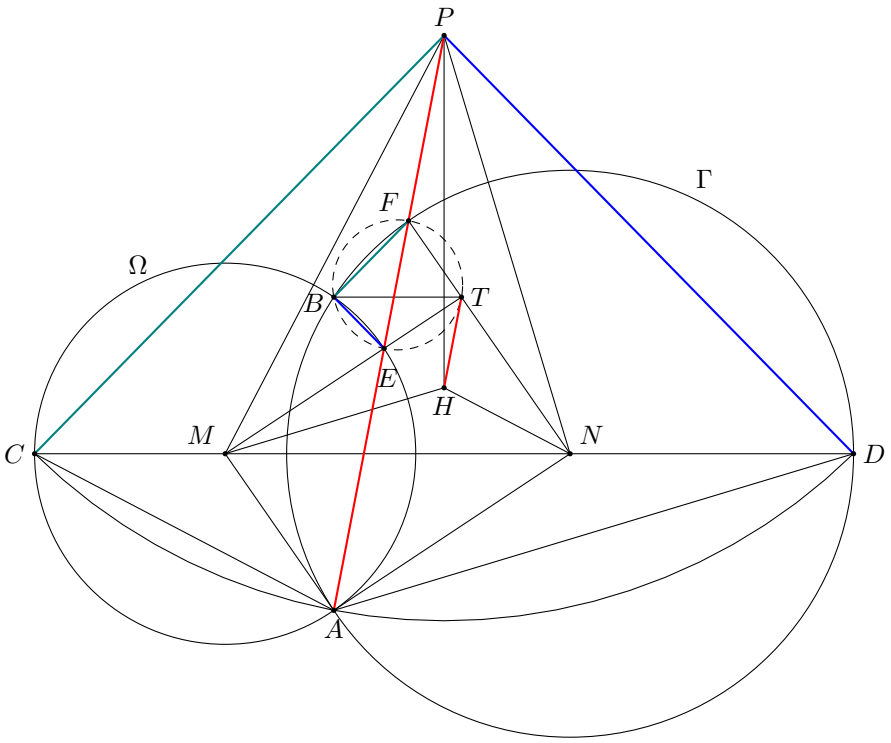
Comment. For an alternative ending, we can employ Pascal's Theorem instead of Desargues' Theorem. Let Y be the point where circle CSR meets the altitude CH. Angle chasing shows that Y lies on BQ and AS. By Pascal's Theorem applied to the hexagon $ASDBQG$, the points X, Y and H are collinear, so X is on the altitude CH.

G4. Let Ω and Γ be circles centred at M and N, respectively, such that the radius of Ω is smaller than the radius of Γ. Suppose circles Ω and Γ intersect at two distinct points A and B. Line MN intersects Ω at C and Γ at D, such that points C, M, N and D lie on the line in that order. Let P be the circumcentre of triangle ACD. Line AP intersects Ω again at $E \neq A$. Line AP intersects Γ again at $F \neq A$. Let H be the orthocentre of triangle PMN.

Prove that the line passing through H parallel to AP is tangent to the circumcircle of triangle BEF.

(Vietnam)

Solution 1. Let T be the intersection of lines ME and NF. We will first show that circle BFE passes through T, then we will show that line TH is parallel to AP and tangent to circle BFE.



We have that $\angle CPA = 2\angle CDA = \angle BDA = \angle BFA$, so we conclude $PC \parallel FB$. Similarly, $PD \parallel BE$. We also have $\angle EFT = \angle NAP = \angle PDN = \angle MCP = \angle PAM = \angle MEA = \angle TEF$, so we get $\triangle EFT$ and $\triangle DCP$ are isosceles and $\triangle EFT \sim \triangle DCP$.

As $\angle FTE = \angle CPD = 180^\circ - \angle EBF$, we have that the points B, F, T, E are concyclic. In circle $BFTE$, we have line BT bisects $\angle EBF$, and we also have that side CD in isosceles triangle CDP is parallel to the external bisector of $\angle CPD$. This means that $BT \parallel CD$.

From $\angle CPH = \frac{1}{2}\angle CPD = \frac{1}{2}(\angle CPA + \angle APD) = \angle MPA + \angle APN = \angle MPN$, we get $\angle CPM = \angle HPN$. Hence,

$$\angle NMT = \angle BTM = \angle BFE = \angle CPA = 2\angle CPM = 2\angle HPN = 2\angle NMH$$

and MH bisects $\angle NMT$. Similarly we get NH bisects $\angle TNM$.

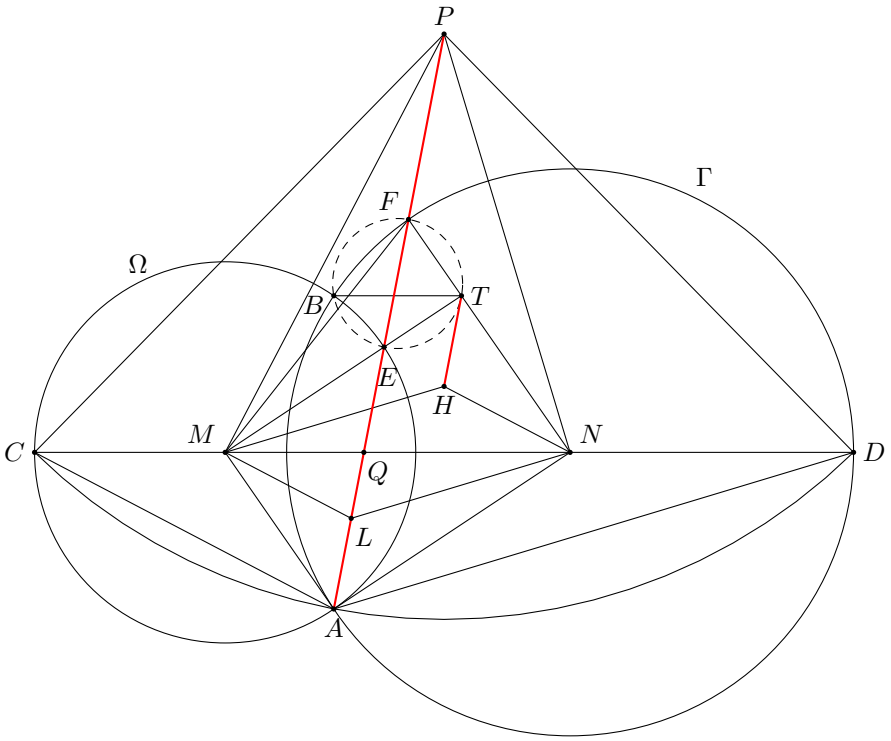
It follows that H is the incentre of $\triangle MTN$, and TH bisects $\angle MTN$. Thus, in isosceles triangle EFT, segment TH is parallel to line $AEFP$ and is tangent to circle EFT.

Solution 2. Observe that AC is a common chord of Ω and circle ACD, thus A and C are symmetric with respect to the line MP. Similarly, A and D are symmetric with respect to the line NP. Since we have $PC = PD$, we have

$$\angle PAM = \angle MCP = \angle PDN = \angle NAP.$$

We now have $\angle MEA = \angle EAM = \angle NAE$, and so $EM \parallel NA$. Similarly, $FN \parallel MA$.

Let T be the intersection of lines ME and NF, so $AMTN$ is a parallelogram.



Since AB is a common chord of Ω and Γ, A and B are symmetric with respect to MN. Hence $\angle NMB = \angle AMN = \angle TNM$ and $BM = AM = NT$, so $MBTN$ is an isosceles trapezium. Hence, we get $\angle TMB = \angle TNB$. We also have $\frac{EM}{MB} = 1 = \frac{FN}{NB}$ and so $\triangle EMB \sim \triangle FNB$. We now have

$$\angle EBF = \angle MBN = \angle MTN = \angle 180^\circ - \angle FTE.$$

Thus, $BFTE$ is cyclic. This means that $\angle EFT = \angle NAF = \angle TEF$, so $\triangle TEF$ is isosceles. Thus it suffices to prove that HT is parallel to EF, as then HT will be tangent to the circumcircle of $\triangle BFE$ at T.

First, we notice that HN is perpendicular to PM and PM is perpendicular to AC. Thus, $HN \parallel AC$. Similarly, $HM \parallel AD$. Let Q be the intersection of lines AP and MN. As $\angle PAM = \angle PDN$, we have that $DAMP$ is cyclic. Similarly, we also have that $CPNA$ is cyclic. Hence, using power of a point we get

$$QC \cdot QN = QA \cdot QP = QM \cdot QD.$$

Thus,

$$\frac{QM}{QC} = \frac{QN}{QD} = k.$$

Let L be the point on segment AQ such that $\frac{QL}{QA} = k$. By similar triangles, we get that $ML \parallel CA \parallel HN$ and $NL \parallel DA \parallel HM$. Thus, $LMHN$ is a parallelogram. Since $AMTN$ is also a parallelogram, we get that HT is parallel to AL, so HT is parallel to EF. This implies that HT is tangent to the circumcircle of $\triangle BFE$ and we are done.

G5. Let $ABCD$ be a convex quadrilateral with $\angle ADC > 90^\circ$. Suppose that points X and Y lie on diagonal AC such that

$$\angle ADX = \angle YDC = 90^\circ, \quad \angle CAD = \angle CBX, \quad \text{and} \quad \angle DCA = \angle YBA.$$

Let O be the circumcentre of triangle BXY. Suppose point $R \neq O$ lies on the perpendicular bisector of AC such that lines OR and AC are parallel.

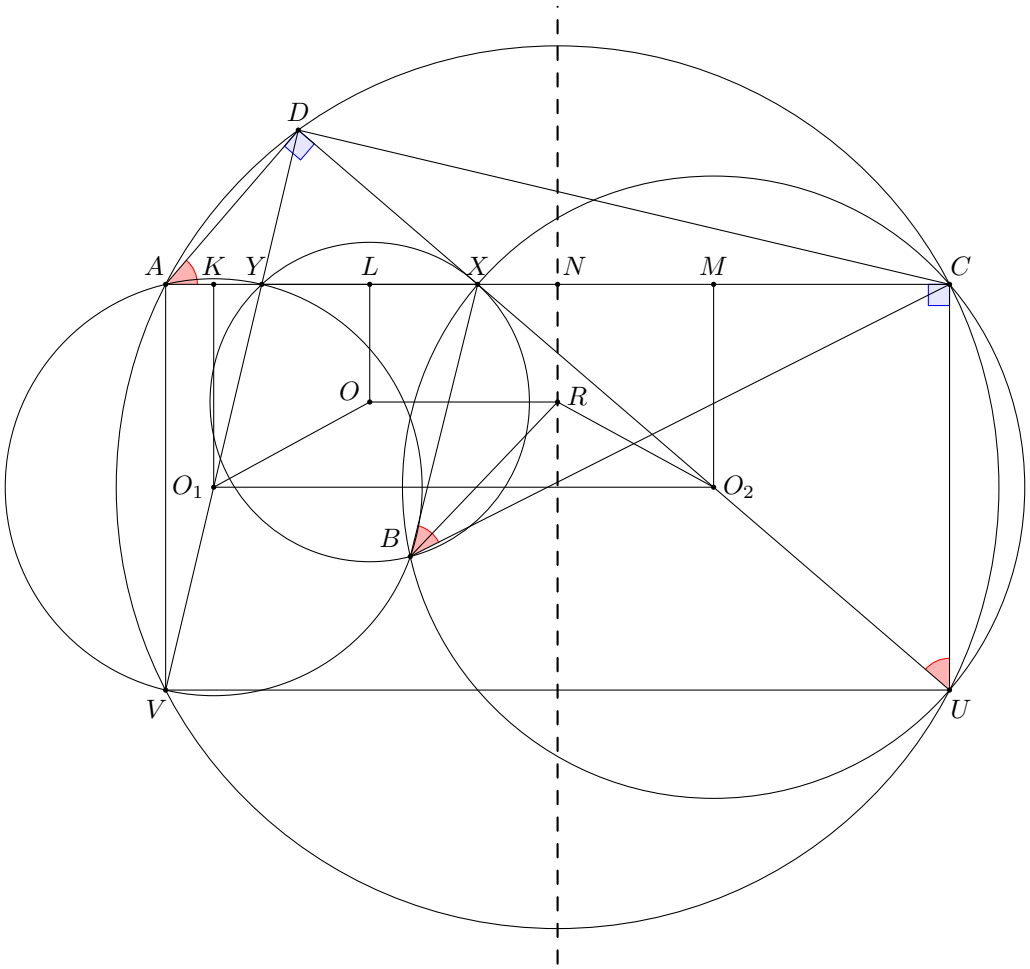
Prove that line RO bisects angle $\angle DRB$.

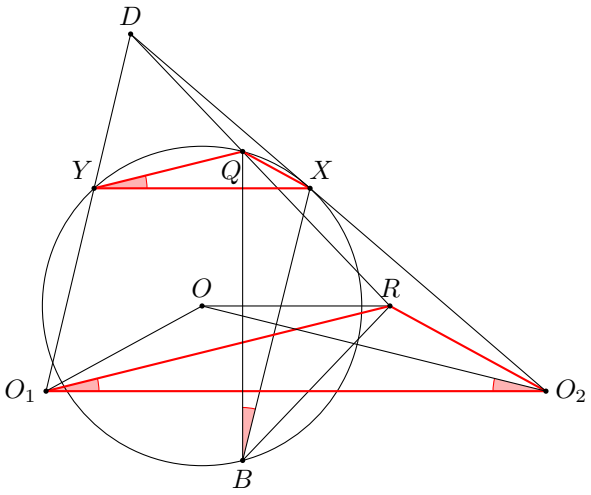
(Romania)

Solution 1. Let Q be the reflection of B in line RO. Since O is the centre of circle BXY, point Q lies on the same circle. Our goal is to show that Q lies on ray RD, as then $\angle DRB$ is equal to $\angle QRB$. This means that RO bisects $\angle DRB$ by the definition of point Q.

By the condition $\angle ADC > 90^\circ$ and $\angle ADX = \angle YDC = 90^\circ$, points A, Y, X and C are distinct and lie on line AC in this order.

Let rays DX and DY meet circle ACD again at $U \neq D$ and $V \neq D$, respectively. Since $\angle ADU = \angle VDC = 90^\circ$, segments AU and CV are diameters in circle ACD, so $AVUC$ is a rectangle. Furthermore, as $\angle CBX = \angle CAD = \angle CUD = \angle CUX$, the quadrilateral $BUCX$ is cyclic. Due to $\angle XCU = \angle ACU = 90^\circ$, segment UX is a diameter in circle $BUCX$. It can be seen similarly that quadrilateral $AVBY$ is cyclic, and VY is a diameter in circle $AVBY$.





Let O_1 and O_2 be the circumcentres of quadrilaterals $AVBY$ and $BUCX$.

Claim 1.1. O_1O_2RO is an isosceles trapezium with $O_1O_2 \parallel RO \parallel AC$.

Proof. Since O_1 and O_2 are the midpoints of VY and UX, respectively, O_1O_2 is the midline of the trapezium $VUXY$, so $O_1O_2 \parallel AC$. On the other hand, RO is perpendicular to the perpendicular bisector of AC, so $RO \parallel AC$. Note that lines O_1O_2 and RO cannot coincide, because $OO_1 \perp BY$, $OO_2 \perp BX$ and points X and Y are distinct.

Let the projections of O_1, O, O_2, R onto line AC be K, L, M, N, respectively; these are the midpoints of AY, YX, XC and AC, respectively. As $\overrightarrow{KL} = \frac{1}{2}\overrightarrow{AX} = \overrightarrow{NM}$, the projections of vectors $\overrightarrow{O_1O}$ and $\overrightarrow{RO_2}$ onto AC are equal, which shows that the trapezium O_1O_2RO is indeed isosceles. □

Claim 1.2. Triangles O_1O_2R and YXQ are similar, and their corresponding sides are parallel.

Proof. We show that $\angle O_2O_1R = \angle XYQ$. Notice that OO_2 is the central line between circles BYX and $BUCX$, so it is perpendicular to the common chord BX. Furthermore, $BQ \perp RO$ and $RO \parallel O_1O_2$. Hence, the sides of angles $\angle OO_2O_1$ and $\angle XBQ$ are perpendicular. Together with the symmetry of the trapezium O_1O_2RO and by angle chasing in circle $BXQY$, we get

$$\angle O_2O_1R = \angle OO_2O_1 = \angle XBQ = \angle XYQ.$$

It can be shown similarly that $\angle RO_2O_1 = \angle QXY$.

Hence, the two triangles are similar with same orientation. As $O_1O_2 \parallel XY$, the other two corresponding pairs of sides are also parallel. □

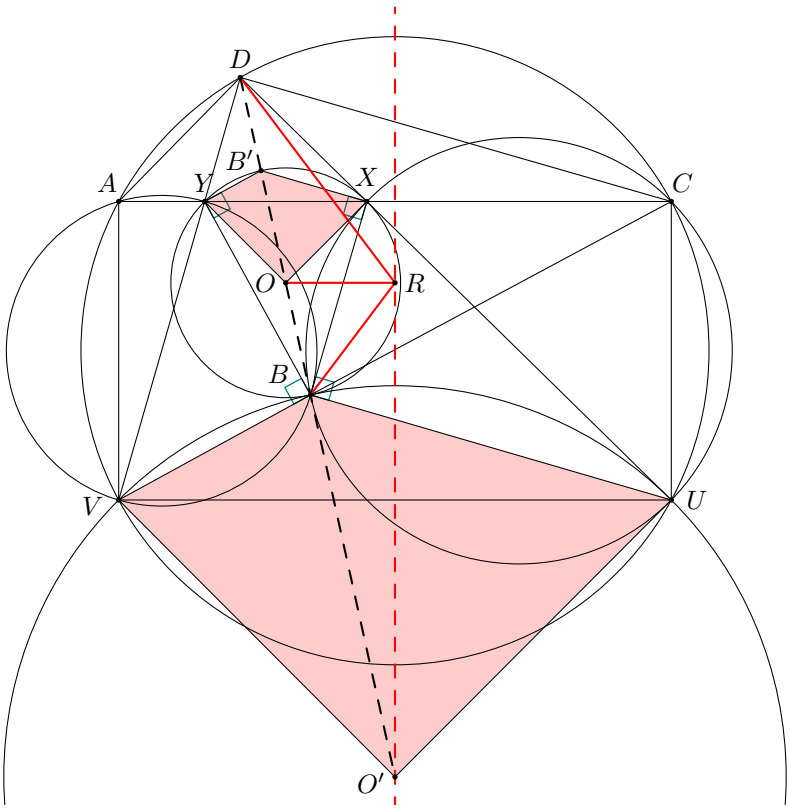
Now, in order to solve the problem, consider the homothety that maps triangle O_1O_2R to triangle YXQ. Points K, Y, X, M lie on segment AC in this order, so $\overrightarrow{O_1O_2} = \overrightarrow{KM}$ and $\overrightarrow{YX}$ have the same direction, and $|O_1O_2| = |KM| > |YX|$. Hence, the ratio of the homothety mapping O_1O_2R to YXQ is positive and less than 1.

Lines O_1Y, O_2X and RQ meet at the homothety centre, so the homothety centre is the intersection of lines O_1Y and O_2X, which is point D. As R is mapped to Q, we conclude that Q lies on segment RD, which completes the solution.

Solution 2. We will prove that lines RD, RB, RO, and the perpendicular bisector of AC form a harmonic bundle.

We construct rectangle $AVUC$ like in the first solution.

On circle BXY, let B' be the point diametrically opposite to B, and let O' be the centre of circle BVU. Note that B cannot lie on line UV, as otherwise both quadrilaterals $AVBY$ and $BUCX$ would be rectangles so X and Y would coincide. By $O'V = O'U$, point O' lies on the common perpendicular bisector of VU and AC.



Claim 2.1. Points D, B', O, B, O' are collinear, and

$$\frac{DO'}{DO} = \frac{BO'}{B'O} \quad \text{(with directed segments).} \tag{1}$$

Proof. Point O is the centre of circle BXY, so O is the midpoint of the diameter BB'.

Segments UX, VY and BB' are diameters in circles $BUCX$, $AVBY$ and $BXB'Y$, respectively, so we have $\angle UBX = \angle YBV = \angle B'XB = \angle BYB' = 90°$. Therefore, both BU and $B'X$ are perpendicular to BX, and both BV and $B'Y$ are perpendicular to BY. From the rectangle $AVUC$ we know that $VU \parallel XY$. Hence, the triangles $B'YX$ and BVU have parallel sides.

Consider the homothety that maps triangle YXB' to VUB. The homothety centre is $UX \cap VY = D$. The homothety maps vertex B' to vertex B, so point D also lies on line BB'. The same homothety maps the circumcentre O of triangle YXB' to the circumcentre O' of triangle VUB. Therefore O' lies on line DO, and hence B, B', O, D and O' are collinear.

Finally, the fractions in equation (1) are equal to the ratio of the homothety. □

Since $RO \perp RO'$, the problem statement is equivalent to the bundle $(RO, RO'; RB, RD)$ being harmonic. By (1) we have

$$\frac{DO}{DO'} = \frac{B'O}{BO'} = -\frac{BO}{BO'},$$

so

$$(RO, RO'; RB, RD) = (O, O'; B, D) = \frac{BO}{BO'} : \frac{DO}{DO'} = -1,$$

as required.

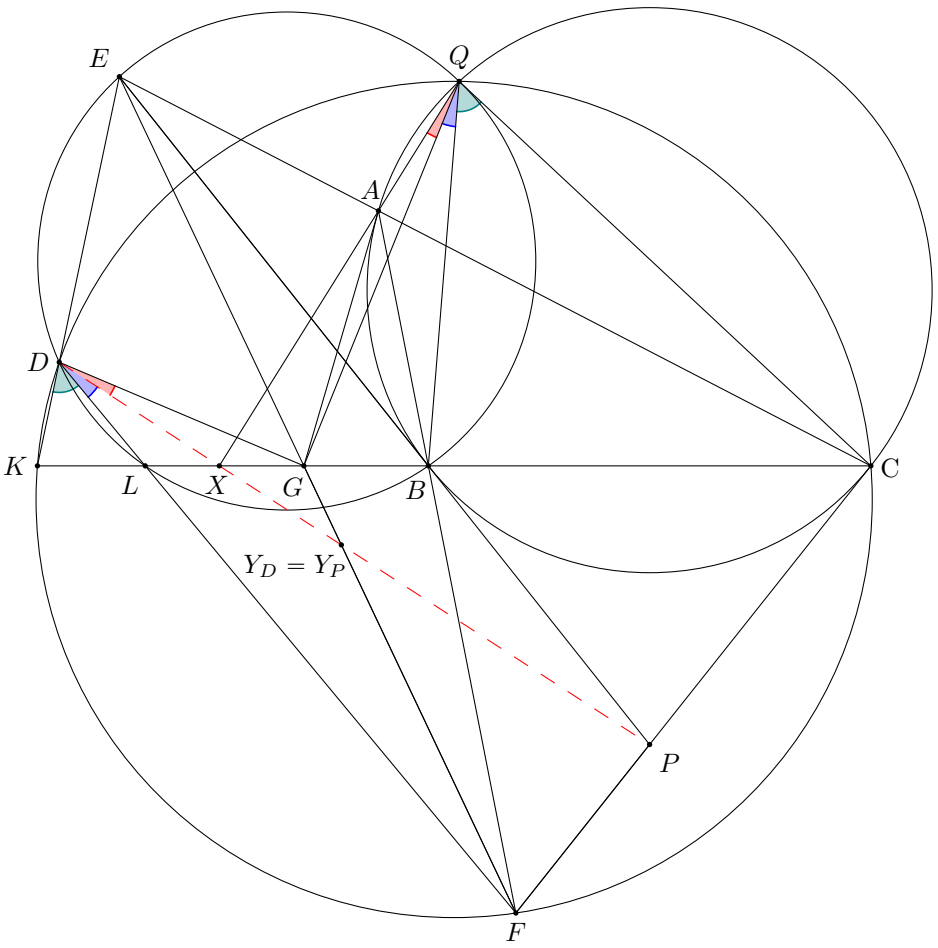
G6. Let ABC be a triangle with $\angle B > \angle A > \angle C$. Let Ω be the circumcircle of triangle ABC. The lines tangent to Ω at points B and C intersect at P. Lines BP and AC intersect at E. Lines CP and AB intersect at F. Let D be the reflection of point A in line EF. The circumcircles of triangles DEB and DFC intersect again at $Q \neq D$. Lines DP and BC intersect at Z.

Prove that points A, Z and Q are collinear.

(India)

Common remarks. All angles in the solutions below are oriented anticlockwise and $\angle ABC$ denotes the angle between lines AB and BC when measured anticlockwise. For two lines ℓ and m, the angle $\angle(\ell, m)$ will denote the anticlockwise angle of rotation that makes ℓ parallel to m.

Solution 1. Let the line tangent to Ω at point A meet line BC at G. By Pascal's Theorem, applied to the degenerate hexagon $AABBCC$, points E, F, G are collinear.



Let $K = BC \cap DE$, $L = BC \cap DF$, and $X = BC \cap AQ$. The idea of this solution is to show the following equality of cross-ratios:

$$(G, X; B, C) = (G, X; K, L). \qquad (*)$$

Then, with points $Y_D = EF \cap DX$ and $Y_P = EF \cap PX$, we have

$$(G, Y_D; E, F) \overset{D}{=} (G, X; K, L) = (G, X; B, C) \overset{P}{=} (G, Y_P, E, F),$$

so $Y_D = Y_P$, and the lines DXY_D and PXY_P coincide. Hence, $X = Z$, which completes the proof.

First, notice that K lies on circle $DCFQ$, and analogously L lies on circle $DBEQ$, because

$$\angle KDF = \angle FAE = \angle BAC = \angle BCP = \angle KCF.$$

Second, Q lies on circle ABC, because

$$\angle BQC = \angle QBL - \angle QCK = \angle QDL - \angle QDK = \angle KDL = \angle FAE = \angle BAC.$$

Third, X is on circle DGQ, because

$$\angle GXQ = \angle CBQ - \angle AQB = \angle CBQ - \angle GAB = \angle LDQ - \angle FDG = \angle GDQ.$$

Now consider the 4-tuples of lines (DG, DX, DL, DK) and (QX, QG, QB, QC). We have

$$\angle(DG, DK) = -\angle(GA, EA) = -\angle(QX, QC),$$

where the second equality holds because GA is tangent to circle AQC. Similarly

$$\angle(DG, DL) = -\angle(QX, QB).$$

Moreover, $\angle(DG, DX) = -\angle(QX, QG)$, since Q, D, X, G are concyclic. These equal angles ensure that (DG, DK, DX, DL) and (QX, QC, QG, QB) are pencils with pairwise equal angles (with opposite orientation). Hence we have equal cross-ratios

$$(G, X; B, C) = (QG, QX; QB, QC) = (DX, DG; DL, DK) = (X, G; L, K) = (G, X; K, L),$$

which completes the proof.

Comment. After proving $QDXG, QDKC$ and $QDLB$ are cyclic quadrilaterals, we can prove the equality of cross-ratios $(G, X; B, C) = (G, X; K, L)$ in a different manner. Let T be the intersection point of lines DQ and BC. By power of point, we obtain $TQ \cdot TD = TB \cdot TL = TC \cdot TK = TX \cdot TG$. Thus, the inversion about a circle with centre T and radius $\sqrt{TQ \cdot TD}$ maps the points G, X, K, L to the points X, G, C, B respectively. Since inversion preserves cross ratios, we obtain $(G, X; K, L) = (X, G; C, B) = (G, X; B, C)$.

Solution 2. Let the tangent to Ω at point A intersect line BC at point G and let lines DE and DF intersect line BC at points X and Y respectively. We define $Q' \neq A$ as the second intersection point of line AZ with Ω. We will prove that Q' lies on the circumcircles of triangles DEB and DFC, and hence $Q' = Q$.

Since line GA is tangent to Ω, we have $\triangle GAB \sim \triangle GCA$, hence

$$\frac{GB}{GC} = \frac{GB}{GA} \cdot \frac{GA}{GC} = \frac{AB^2}{AC^2}.$$

Similarly, $EA/EC = BA^2/BC^2$ and $FB/FA = CB^2/CA^2$. By the converse of Menelaus' Theorem for triangle ABC we conclude that the points E, F, G are collinear.

Let line AP intersect Ω at point $K \neq A$. Since G lies on the polar of P in Ω, P lies on the polar of G in Ω. Hence line AP is the polar of G in Ω, so line GK is tangent to Ω at point K. Let line GQ' intersect Ω at point $R \neq Q'$. Then

$$\frac{RA^2}{Q'A^2} = \frac{RG}{Q'G} = \frac{RK^2}{Q'K^2},$$

so $ARKQ'$ is a harmonic quadrilateral in Ω, that is $(A, K; Q', R)_\Omega = -1$. Similarly, $(C, B; A, K)_\Omega = -1$. Let lines BC and EF intersect line AP at points L and J respectively. Let lines DP and

EF intersect at point V. Let lines AG and AR intersect line DP at points S and V' respectively. Projecting, we have

$$(S,P;Z,V')_{DP} \overset{A}{=} (A,K;Q',R)_{\Omega} = -1$$

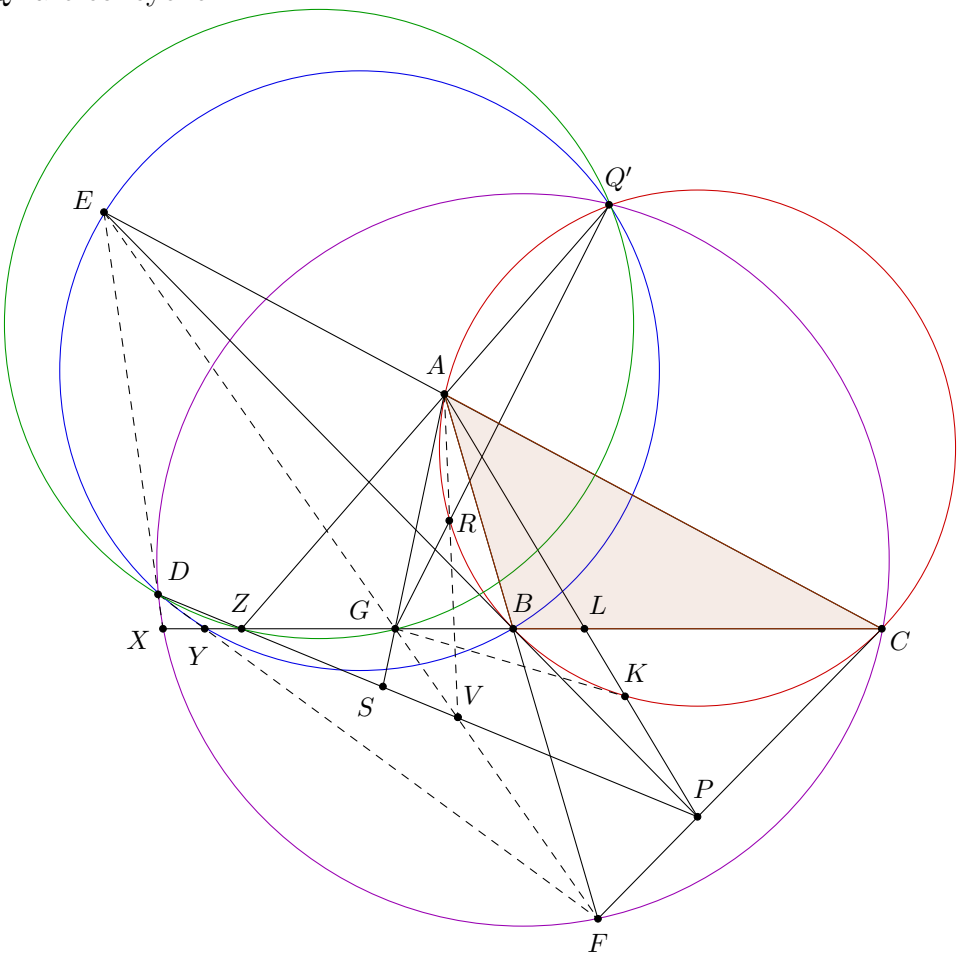
and

$$(S,P;Z,V)_{DP} \overset{G}{=} (A,P;L,J)_{AP} \overset{C}{=} (E,F;G,J)_{EF} \overset{A}{=} (C,B;A,K)_{\Omega} = -1$$

hence $V = V'$, so A, R, V are collinear. Now

$$\angle ZDG = \angle VDG = \angle GAV = \angle GAR = \angle AQ'G = \angle ZQ'G$$

so D, Z, G, Q' are concyclic.



We will now show that $BYDE$ and $CFXD$ are cyclic quadrilaterals. Note that

$$\angle DYB = \angle FYB = \angle FBY - \angle BFY = \angle ABC - 2\angle AFE$$

and

$$\angle DEB = \angle DEA - \angle BEA = 2\angle FEA - (\angle BAC - \angle ACB)$$

so since $\angle FEA + \angle AFE = \angle FAE = \angle BAC$ and $\angle ABC = \angle BAC + \angle ACB$, we conclude $\angle DYB = \angle DEB$. Thus $BYDE$ is cyclic, and likewise, $CFXD$ is cyclic. Finally,

$$\begin{aligned}\angle DQ'B &= \angle DQ'Z + \angle ZQ'B = \angle DGZ + \angle AQ'B \\ &= \angle DGZ + \angle GAB = \angle DGZ + \angle FDG = \angle DYB,\end{aligned}$$

so $BYDQ'$ is cyclic. Likewise, $CXDQ'$ is cyclic. Together with the fact that $BYDE$ and $CFXD$ are cyclic quadrilaterals, we conclude that $Q' = Q$, so Q lies on line AZ. This concludes the proof.

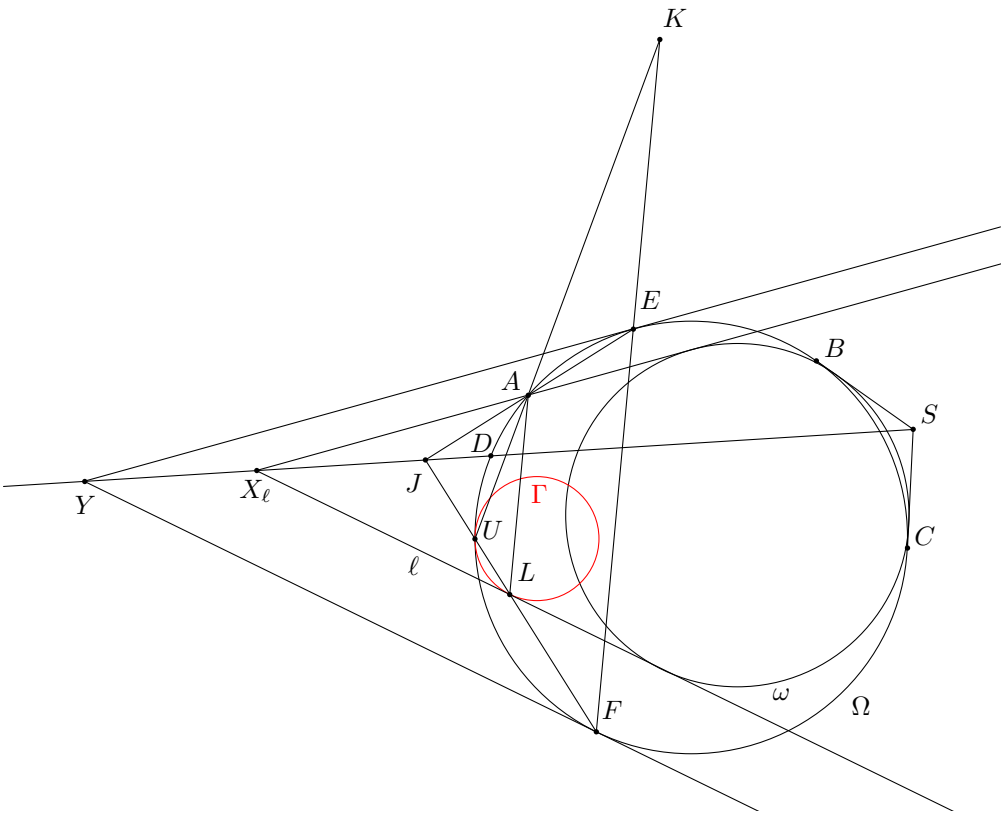
G7. Let $ABCD$ be a cyclic quadrilateral with circumcircle Ω such that $CD > BD > BC$. The tangents to Ω at B and C meet at a point S. A line ℓ is called *coastal* if:

- ℓ does not pass through A,
- ℓ intersects ray SD at a point X_ℓ beyond D, and
- there is a circle tangent to line ℓ, line $X_\ell A$, segment SB and segment SC.

Prove that there is a circle that is tangent to Ω and all coastal lines.

(Islamic Republic of Iran)

Solution 1. Let K be the pole of SD with respect to Ω, and let U be the second intersection of KA with Ω.

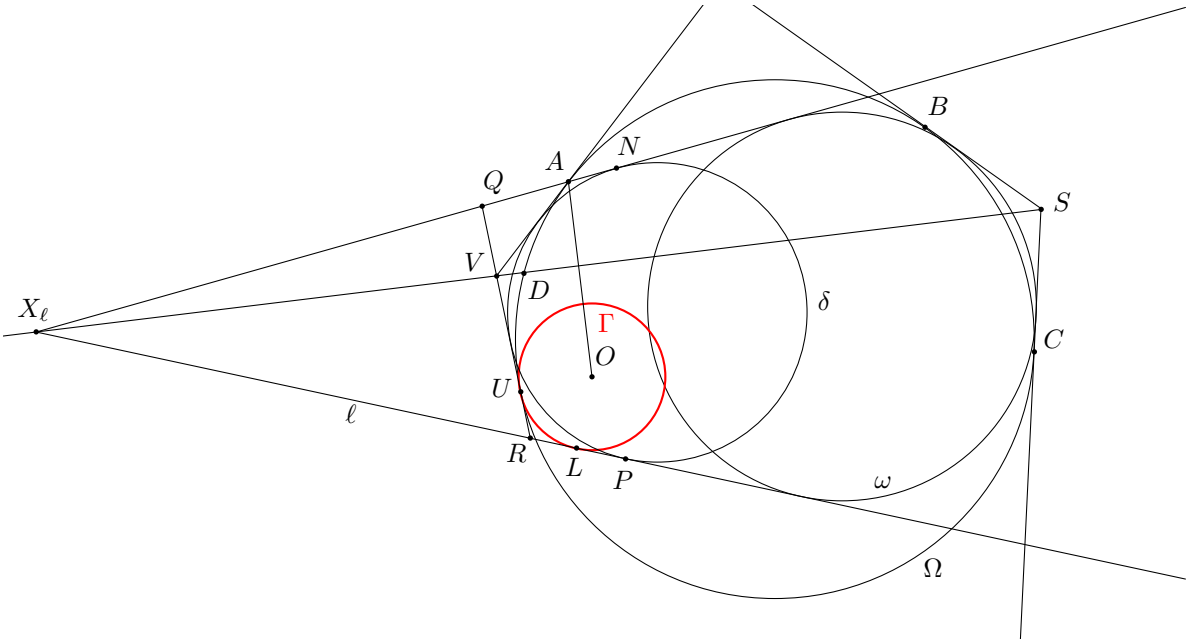


Consider a coastal line ℓ. Let ω denote the circle tangent to ℓ, $X_\ell A$, SB and SC. Since ω is tangent to segments SB and SC, there is a dilation ρ centred at S sending ω to Ω. Let Y be the image of X_ℓ under ρ. Then the images of $X_\ell A$ and ℓ under ρ are tangent to Ω, so let these touch Ω at E and F respectively.

Since K is the pole of $X_\ell D$, it lies on the polar EF of Y. Hence the intersection J of AE and UF lies on $X_\ell D$ since it is the polar of K. Let UF intersect ℓ at L. Let ν be the dilation centred at U sending F to L and let Γ be the image of Ω under this dilation. Then ℓ touches Γ at L and Ω touches Γ at U. Hence it suffices to show that Γ is a fixed circle, that is, that $\frac{UL}{UF}$ is constant. Indeed, by the parallel lines, there is a dilation around J sending triangle $X_\ell AL$ to triangle YEF, so AL is parallel to EF. Then $\frac{UL}{UF} = \frac{UA}{UK}$, which is constant.

Solution 2. We will find a fixed point U on Ω and a circle Γ (not dependent on ℓ) such that Ω touches Γ at U and ℓ is tangent to Γ.

Let the tangent to Ω at A meet line SD at V and let the other tangent from V touch Ω at U. Let Γ be the circle which is tangent to Ω at U such that the centre O of Γ lies on the perpendicular from A to SD. Let r denote the radical axis of Γ and the point circle A. Since $VU = VA$ it follows that V is on r. Also, r must be perpendicular to AO, so SD is the radical axis of Γ and the point circle A.



Fix a coastal line ℓ. Let ω denote the circle which is tangent to ℓ, $X_\ell A$, SB and SC. Let VU intersect $X_\ell A$ and ℓ at Q and R respectively.

Claim. $X_\ell A + AV = X_\ell R + RV$.

Proof. Let δ be the excircle opposite X_ℓ of triangle $X_\ell QR$. By Monge's Theorem applied to δ, ω and Ω, VA is tangent to δ. Hence by Pitot's Theorem, $X_\ell A + AV = X_\ell R + RV$. □

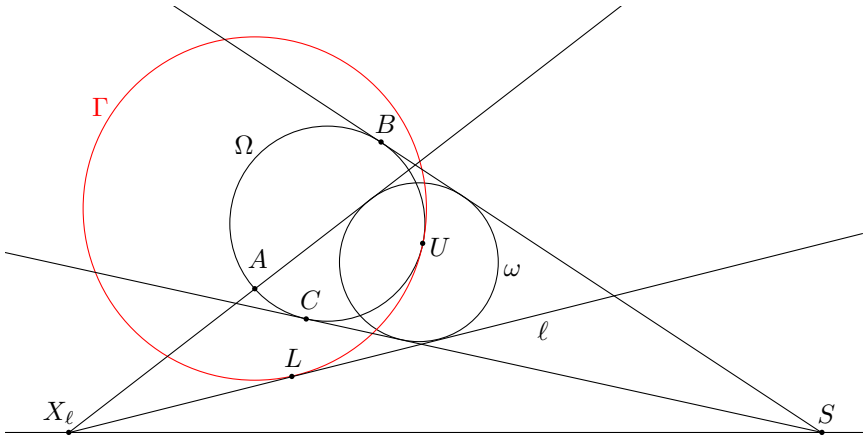
Let L be the point on ray $X_\ell R$ such that $X_\ell L = X_\ell A$. Then

$$RL = X_\ell L - X_\ell R = X_\ell A - (X_\ell A + AV - RV) = RV - UV = RU.$$

Hence, there is a circle Γ_ℓ tangent to RL at L and RU at U. Since $X_\ell L = X_\ell A$, X_ℓ is on the radical axis of Γ_ℓ and the point A. If Γ_ℓ and Γ are the same circle then ℓ is tangent to Γ, as required. If they are different circles, then by the Radical Axis Theorem between these circles and the point circle A, the point X_ℓ lies on the radical axis VU of Γ_ℓ and Γ, hence $X_\ell = V$. But then the circle tangent to line $X_\ell A$ and segments SB and SC is $\omega = \Omega$, which means that ℓ is the other tangent VU from $X_\ell = V$ to Ω. This means that ℓ is tangent to Γ, as required.

Comment. Solution 2 can be adapted to a generalisation of the problem where A is replaced by a circle internally tangent to Ω and the line $X_\ell A$ is replaced by a line through X_ℓ tangent to this circle.

Comment. If we modify the diagram in such a way that all points and circles are on the same side of SX_ℓ, then we can interpret it as a picture in the hyperbolic half-plane model, where SX_ℓ is the horizontal axis. Then the problem can be solved as follows:



The rays SB and SC are hypercycles with same ideal points, so the (hyperbolic) distance between them is some constant d. Then the two circles Ω and ω inscribed between these lines have diameter d. Similarly, the distance between hypercycles $X_\ell A$ and ℓ is equal to the diameter of ω, which is d. Finally, the distance between ℓ and A is d, so ℓ is tangent to the circle Γ of (hyperbolic) radius d around A. Since Ω has (hyperbolic) diameter d, and A is on Ω, it follows that Γ is tangent to Ω.

This solution can be rephrased in more elementary terms, for example using cross ratios or trigonometry to define the hyperbolic distance between hypercycles (which are lines in this problem). Moreover, these elementary versions of the solutions can apply directly to the problem as stated, where line SX_ℓ intersects with circle Ω.

Uses of hyperbolic geometry to solve Olympiad-style problems have been discussed for example in G. Kós: *Térbe kilépő bizonyítások VI – Állandó távolságú görbepárok* (3D lifting proofs VI – Curve pairs with constant distance), KöMaL 70/3 (March 2020), pp. 130–141, in Hungarian, available at https://epa.oszk.hu/03400/03409/00025/pdf/EPA03409_komal_2020_03_130-141.pdf.

Number Theory

N1. Let n and k be positive integers such that $n < k < 2n$. Suppose $a_1, a_2, \ldots, a_k$ are positive integers such that $2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}$ is divisible by $2^n - 1$.

Show that at least three of $a_1, a_2, \ldots, a_k$ have the same remainder when divided by n.

(Mongolia)

Solution 1. Since $2^{a+n} \equiv 2^a \pmod{2^n - 1}$, the conditions and conclusion of the problem are thus unchanged when each a_j is replaced with another positive integer congruent to a_j mod n. So we may assume that each a_j is contained in $\{0, 1, 2, \ldots, n-1\}$, and it suffices to show that three of the a_j are equal.

Suppose for contradiction that no three of $a_1, a_2, \ldots, a_k$ are equal. Then

$$0 < 2^{a_1} + 2^{a_2} + \cdots 2^{a_k} \leqslant 2(1 + 2 + \cdots + 2^{n-1}) = 2(2^n - 1). \tag{1}$$

If the second inequality were an equality, then each element of $\{0, \ldots, n-1\}$ would appear twice among $a_1, \ldots, a_k$, contradicting $k < 2n$. Thus both inequalities of (1) are strict.

Since $2^n - 1$ divides $2^{a_1} + 2^{a_2} + \cdots 2^{a_k}$, we then have

$$2^{a_1} + 2^{a_2} + \cdots 2^{a_k} = 2^n - 1 \tag{2}$$

For $i \in \{0, \ldots, n-1\}$, let c_i be the number of a_j equal to i. Then $c_0 + \cdots + c_{n-1} = k$, and we have assumed all $c_i \leqslant 2$. Equation (2) then becomes

$$c_0 2^0 + c_1 2^1 + \cdots + c_{n-1} 2^{n-1} = 2^n - 1. \tag{3}$$

We prove by strong induction that $c_i = 1$ for all i. As $2^n - 1$ is odd, considering equation (3) mod 2 implies that c_0 is odd. As $0 \leqslant c_0 \leqslant 2$, we have $c_0 = 1$. Suppose that $c_0 = \cdots = c_{s-1} = 1$ for some $1 \leqslant s \leqslant n-1$. The first s terms of equation (3) then sum to $2^s - 1$, and reducing mod 2^{s+1} then yields

$$2^s - 1 + c_s 2^s \equiv -1 \mod 2^{s+1}, \quad \text{which implies} \quad 2^{s+1} \mid 2^s(1 + c_s).$$

Thus c_s is odd, but as $0 \leqslant c_s \leqslant 2$ then $c_s = 1$. By induction then all $c_i = 1$ as claimed.

Therefore, exactly one of $a_1, a_2, \ldots, a_k$ is equal to i, for each $i \in \{0, \ldots, n-1\}$, and we have $k = c_0 + \cdots + c_{n-1} = n$, contradicting $k > n$.

This yields the desired contradiction, and three of $a_1, \ldots, a_k$ are equal.

Comment. The strong induction above can also be achieved by subtracting the first s terms from the left hand side of equation (3) and then dividing by 2^s. The first s terms sum to $2^s - 1$ so this yields

$$c_s + c_{s+1} 2^1 + \cdots + c_{n-1} 2^{n-s-1} = 2^{n-s} - 1.$$

As $n - s \geqslant 1$, the right hand side is odd, so considering this equation mod 2 yields $c_s = 1$. By induction then all $c_i = 1$ as claimed.

Solution 2. First note that for nonnegative integers a, b, we have $2^a \equiv 2^b \bmod 2^n - 1$ if and only if $a \equiv b \bmod n$. Indeed, if $a = jn + b$ for $j \in \mathbb{Z}$ then $2^a = (2^n)^j \cdot 2^b \equiv 2^b \bmod 2^n - 1$; and $2^0, 2^1, \ldots, 2^{n-1}$ are distinct mod $2^n - 1$.

Suppose for a contradiction that for each $i \in \{0, \ldots, n-1\}$, there are at most two $j \in \{1, \ldots, k\}$ such that $a_j \equiv i \bmod n$. For $r = 0, 1, 2$, let B_r be the set of $i \in \{0, \ldots, n-1\}$ such that precisely r of the a_j satisfy $a_j \equiv i \bmod n$, or equivalently, $2^{a_j} \equiv 2^i \bmod 2^n - 1$. Let $b_r = |B_r|$.

The B_r form a partition of $\{0,\dots,n-1\}$, so $b_0+b_1+b_2=n$. For each $j\in\{1,\dots,k\}$, a_j is counted in precisely one of the B_r, and each B_r counts rb_r of the a_j, so $b_1+2b_2=k$. Subtracting these two equations gives $b_2-b_0=k-n$, so from the given inequalities, $0<b_2-b_0<n$.

The given condition $2^{a_1}+\cdots+2^{a_k}\equiv 0 \bmod 2^n-1$ is then equivalent to

$$\sum_{i\in B_1} 2^i+\sum_{i\in B_2} 2\cdot 2^i\equiv 0 \mod 2^n-1. \tag{4}$$

On the other hand, as the B_r form a partition of $\{0,\dots,n-1\}$,

$$\sum_{i\in B_0} 2^i+\sum_{i\in B_1} 2^i+\sum_{i\in B_2} 2^i\equiv 2^0+\cdots+2^{n-1}\equiv 0 \mod 2^n-1. \tag{5}$$

Subtracting (4) from (5) yields

$$\sum_{i\in B_0} 2^i\equiv\sum_{i\in B_2} 2^i \mod 2^n-1. \tag{6}$$

As B_0 and B_2 are subsets of $\{0,\dots,n-1\}$, the left and right sums of (6) are both at most $2^0+\cdots+2^{n-1}=2^n-1$. Thus the two sums are equal, or they are 0 and 2^n-1 in some order.

If one sum of equation (6) is 0 and the other is 2^n-1, then B_0,B_2 are $\varnothing$ and $\{0,\dots,n-1\}$ in some order, so $b_2-b_0=\pm n$, contradicting $0<b_2-b_0<n$. So the two sums are equal.

The two sums of equation (6) then yield binary representations of the same integer, so $B_0=B_2$. As B_0,B_2 are disjoint then both are empty, so $b_0=b_2=0$, contradicting $b_2-b_0>0$.

This yields the desired contradiction, and three of $a_1,\dots,a_k$ are congruent modulo n.

Solution 3. Denote by $\langle a\rangle_n$ the remainder when a is divided by n, that is, the number in $\{0,\dots,n-1\}$ such that $\langle a\rangle_n\equiv a \bmod n$.

As in Solution 1, $2^a\equiv 2^b \bmod 2^n-1$ if and only if $a\equiv b \bmod n$. In particular, $2^{\langle a\rangle_n}\equiv 2^a \bmod 2^n-1$.

For a nonnegative integer t, define

$$f(t)=2^{\langle a_1+t\rangle_n}+2^{\langle a_2+t\rangle_n}+\cdots+2^{\langle a_k+t\rangle_n}$$

so $f(t)\equiv 2^t(2^{a_1}+\cdots+2^{a_k})\equiv 0 \bmod 2^n-1$. As t varies over $\{0,1,\dots,n-1\}$, $\langle a_i+t\rangle_n$ varies over the same set, so

$$\sum_{t=0}^{n-1} 2^{\langle a_i+t\rangle_n}=2^0+2^1+\cdots+2^{n-1}=2^n-1.$$

We then have

$$\sum_{t=0}^{n-1} f(t)=\sum_{t=0}^{n-1}(2^{\langle a_1+t\rangle_n}+2^{\langle a_2+t\rangle_n}+\cdots+2^{\langle a_k+t\rangle_n})=k\,(2^n-1)\,.$$

As each $f(t)$ is a positive multiple of 2^n-1, and $k<2n$, there exists $t\in\{0,\dots,n-1\}$ such that $f(t)=2^n-1$. For this t, and for each $i\in\{0,\dots,n-1\}$, let c_i be the number of a_j such that $\langle a_j+t\rangle_n=i$. Then $c_0+\cdots+c_{n-1}=k$, and

$$f(t)=c_0 2^0+c_1 2^1+\cdots+c_{n-1}2^{n-1}=2^n-1. \tag{7}$$

We now proceed as in Solution 1.

N2. In an $n \times n$ board, for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, the cell in the i^{th} row from the bottom and j^{th} column from the left contains $\gcd(i, j)$ leaves. A koala travels from the bottom left corner to the top right corner. At each step, the koala moves up or to the right by a single cell. The koala eats the leaves in each cell it visits, including the bottom left and top right cells.

Determine the maximum number of leaves that the koala can eat.

Here $\gcd(x, y)$ *denotes the greatest common divisor of integers* x *and* y.

(India)

Answer: The maximum number of leaves is $\frac{n^2+3n-2}{2}$.

Solution 1. By alternating between upward and rightward steps, the koala visits the n diagonal cells, eating $1+2+\cdots+n$ leaves, and the $n-1$ cells immediately above the diagonal, eating one leaf in each. The koala eats a total of

$$(1+2+\cdots+n)+(n-1)=\frac{n(n+1)}{2}+n-1=\frac{n^2+3n-2}{2}$$

leaves.

We now show that the koala cannot eat any more leaves. Consider the numbers $\gcd(i, j)$ and $\gcd(i+1, j)$. They divide i and $i+1$ respectively, but i and $i+1$ are coprime, hence $\gcd(i, j)$ and $\gcd(i+1, j)$ are coprime. As $\gcd(i, j)$ and $\gcd(i+1, j)$ are coprime and both divide j, their product divides j, hence $\gcd(i+1, j)\gcd(i, j) \leqslant j$.

The function $f(x) = x + \frac{j}{x}$ for $x \in [1, j]$ is maximised when $x = 1$ or $x = j$. Indeed,

$$f(x)-(j+1)=(x-1)+\left(\frac{j}{x}-j\right)=(x-1)-\frac{j}{x}\cdot(x-1)=\frac{(x-1)(x-j)}{x}\leqslant 0$$

with equality if and only if $x = 1$ or $x = j$. So we have

$$\gcd(i, j)+\gcd(i+1, j)\leqslant \gcd(i, j)+\frac{j}{\gcd(i, j)}\leqslant j+1.$$

Similarly, $\gcd(i, j)+\gcd(i, j+1)\leqslant i+1$. Letting $E(i, j) = ij$ these inequalities become

$$\begin{aligned}\gcd(i, j)+\gcd(i+1, j)&\leqslant E(i+1, j)-E(i, j)+1,\\ \gcd(i, j)+\gcd(i, j+1)&\leqslant E(i, j+1)-E(i, j)+1.\end{aligned}$$

Suppose the koala visits cells $(1, 1) = (i_1, j_1), (i_2, j_2), \ldots, (i_{2n-1}, j_{2n-1}) = (n, n)$, in order. At cell (i_t, j_t), the koala eats $\gcd(i_t, j_t)$ leaves. The above inequalities imply

$$\gcd(i_t, j_t)+\gcd(i_{t+1}, j_{t+1})\leqslant E(i_{t+1}, j_{t+1})-E(i_t, j_t)+1,$$

hence

$$\begin{aligned}\sum_{t=1}^{2n-1}\gcd(i_t, j_t)&=\frac{\gcd(1,1)+\gcd(n,n)}{2}+\frac{1}{2}\sum_{t=1}^{2n-2}\left(\gcd(i_t, j_t)+\gcd(i_{t+1}, j_{t+1})\right)\\ &\leqslant\frac{n+1}{2}+\frac{1}{2}\sum_{t=1}^{2n-2}\left(E(i_{t+1}, j_{t+1})-E(i_t, j_t)+1\right)\\ &=\frac{n+1}{2}+\frac{2n-2}{2}+\frac{E(n,n)-E(1,1)}{2}\\ &=\frac{n^2+3n-2}{2}\end{aligned}$$

as required.

Comment. The upper bound can be proved using $\gcd(i,j) + \gcd(i+1,j) \leqslant j+1$ and $\gcd(i,j) + \gcd(i,j+1) \leqslant i+1$ without using the function E. Award the koala a *score* of $i+1$ if it moves right along row i, and a score of $j+1$ if it moves up in column j. Letting S be the total score, the number of leaves eaten is at most $(S+n+1)/2$, since every cell is counted twice except the first and last cells (which are counted once).

Starting from (i,j), an up move followed by a right move gets the same score of $i+j+2$ as a right move followed by an up move. All possible paths of the koala are related by such changes, so the total score is independent of the koala's path; it is n^2+2n-3.

Solution 2. After showing the koala can eat $\frac{n^2+3n-2}{2}$ leaves as in Solution 1, we now show the koala cannot eat any more leaves.

Say that the koala starts at time $t=2$ in cell $(i_2, j_2) = (1,1)$, stepping to cell (i_t, j_t) at time t. Each step is up or to the right by a single cell, so $i_t + j_t = t$ for all integers $2 \leqslant t \leqslant 2n$. Let $g_t = \gcd(i_t, j_t)$ be the number of leaves eaten at time t.

As $i_t + j_t = t$, we have $g_t = \gcd(t, j_t)$. So g_t is a divisor of t, and as $j_t < t$, we observe g_t is a proper divisor of t. Hence $g_t \leqslant t/2$.

We now show, for integers k such that $2 \leqslant k \leqslant n$, the inequality

$$g_{2k-1} + g_{2k} \leqslant k+1. \tag{1}$$

Suppose for contradiction that $g_{2k-1} + g_{2k} \geqslant k+2$. Using $g_t \leqslant t/2$ we have $g_{2k} \leqslant k$ and $g_{2k-1} \leqslant k$, so $g_{2k-1} \geqslant 2$ and $g_{2k} \geqslant 2$.

Without loss of generality assume that the move at time $2k$ is upwards, so $i_{2k} = i_{2k-1} + 1$ and $j_{2k-1} = j_{2k}$. As in Solution 1, $\gcd(i,j)\gcd(i+1,j) \leqslant j$, so $g_{2k-1}g_{2k} \leqslant j_{2k}$. Now expanding

$$(g_{2k-1}-2)(g_{2k}-2) \geqslant 0 \quad \text{yields} \quad g_{2k-1}g_{2k} \geqslant 2\,(g_{2k-1}+g_{2k}) - 4.$$

Using $g_{2k-1}g_{2k} \leqslant j_{2k}$ and our assumption $g_{2k-1} + g_{2k} \geqslant k+2$, we obtain $j_{2k} \geqslant 2k$. This is a contradiction as $j_t < t$ for all t, proving (1).

Now we obtain the desired inequality:

$$\sum_{t=2}^{2n} g_t = g_2 + \sum_{k=2}^{n} (g_{2k-1} + g_{2k}) \leqslant 1 + \sum_{k=2}^{n} (k+1) = \frac{n^2+3n-2}{2}.$$

Comment. The inequality (1) also applies to each pair g_{2k}, g_{2k+1}, and a similar argument shows $g_{2k} + g_{2k+1} \leqslant k+1$. In fact one can show by a similar argument that, in general,

$$\gcd(i,j) + \gcd(i+1,j) \leqslant \left\lceil \frac{i+j}{2} \right\rceil + 1.$$

This inequality can also be proven using similar arguments to Solution 1. If one of the gcds equals 1, we can use $\gcd(a,b) \leqslant (a+b)/2$. Otherwise, we can use the fact that $x + j/x$ is maximised when $x = 2$ or $j/2$ to get $\gcd(i,j) + \gcd(i+1,j) \leqslant j/2 + 2$.

N3. An infinite sequence $a_1, a_2, \ldots$ consists of positive integers, each of which has at least four positive divisors. For each $n \geqslant 1$, a_{n+1} is the sum of the three largest proper divisors of a_n.

Determine all possible values of a_1.

A proper divisor *of a positive integer N is a divisor of N other than N.*

(Lithuania)

Answer: The initial term a_1 can be any positive integer of the form $a_1 = 12^M \cdot 6N$, where M and N are nonnegative integers such that $\gcd(N, 10) = 1$.

Solution. Throughout, by divisors we mean positive divisors. Let $d_i(a)$ be the i^{th} largest proper divisor of the positive integer a.

First we show each term of the sequence is even. Otherwise, if a_n is odd for some n, then

$$a_{n+1} = d_1(a_n) + d_2(a_n) + d_3(a_n) \leqslant \frac{a_n}{3} + \frac{a_n}{5} + \frac{a_n}{7} < a_n.$$

Since a_{n+1} is the sum of three odd divisors, it is also odd. It follows that we have an infinite strictly decreasing sequence of positive integers, providing a contradiction.

Next, we show that if a_n is not divisible by 3, then a_{n+1} is not divisible by 3. We consider two cases: a_n divisible by 4 or not. If a_n is divisible by 4, then

$$a_{n+1} = \frac{a_n}{2} + \frac{a_n}{4} + d_3(a_n) = 3 \cdot \frac{a_n}{4} + d_3(a_n) \equiv d_3(a_n) \not\equiv 0 \pmod 3.$$

On the other hand, if a_n is not divisible by 4, let p be the smallest odd prime divisor of a_n. Then $d_1(a_n) = \frac{a_n}{2}$ is odd, $d_2(a_n) = \frac{a_n}{p}$ is even, and $d_3(a_n) = a_{n+1} - d_1(a_n) - d_2(a_n)$ is odd. Then $\frac{a_n}{d_3(a_n)}$ must be the second smallest even divisor of a_n, so must be $2p$, hence

$$a_{n+1} = \frac{a_n}{2} + \frac{a_n}{p} + \frac{a_n}{2p} = 3 \cdot \frac{a_n}{2p} + \frac{a_n}{2} \equiv \frac{a_n}{2} \not\equiv 0 \pmod 3.$$

Thus, if $a_n \not\equiv 0 \pmod 3$ for some n, then $a_{n+1} \not\equiv 0 \pmod 3$. In this case we have

$$a_{n+1} = d_1(a_n) + d_2(a_n) + d_3(a_n) \leqslant \frac{a_n}{2} + \frac{a_n}{4} + \frac{a_n}{5} < a_n$$

so we have an infinite strictly decreasing sequence of positive integers, a contradiction.

We have now shown that each term of the sequence is divisible by 6. Therefore

$$d_1(a_n) = \frac{a_n}{2},\ d_2(a_n) = \frac{a_n}{3},\ d_3(a_n) = \frac{a_n}{4} \text{ or } \frac{a_n}{5} \text{ or } \frac{a_n}{6},\ \forall n = 1, 2, 3, \cdots.$$

There are three cases for each a_n, as follows.

(i) If a_n is divisible by 4 then $a_{n+1} = \frac{a_n}{2} + \frac{a_n}{3} + \frac{a_n}{4} = \frac{13}{12} \cdot a_n$.

(ii) If a_n is divisible by 5 but not divisible by 4, then $a_{n+1} = \frac{a_n}{2} + \frac{a_n}{3} + \frac{a_n}{5} = \frac{31}{30} \cdot a_n$. However in this case a_{n+1} is odd, so this case in fact never happens.

(iii) If a_n is not divisible by 4 or 5, then $a_{n+1} = \frac{a_n}{2} + \frac{a_n}{3} + \frac{a_n}{6} = a_n$.

Note that in all cases $a_{n+1} \geqslant a_n$ and $\nu_2(a_{n+1}) \leqslant \nu_2(a_n)$. As $\nu_2(a_n)$ is a non-increasing sequence of nonnegative integers, it must be eventually constant. As $\nu_2(a_{n+1}) = \nu_2(a_n)$ only in case (iii), then eventually only case (iii) applies. Thus the sequence a_n is eventually constant, and there exists L such that $a_L = a_{L+1} = \cdots$. The eventual value a_L is divisible by 6, but not divisible by 4 or 5, so $a_L = 6K$ where $K > 0$ is coprime with 10.

Choosing L to be minimal, we have $\nu_2(a_1) > \nu_2(a_2) > \cdots > \nu_2(a_L) = 1$. So for $1 \leqslant n \leqslant L-1$, case (i) applies, a_n is divisible by 4 and $a_{n+1} = \frac{13}{12} \cdot a_n$. Therefore $a_1 = 12^{L-1} \cdot \frac{a_L}{13^{L-1}}$ is of the form $a_1 = 12^M \cdot 6 \cdot N$, where $M \geqslant 0$, $N > 0$ and $N = K/13^{L-1}$ is coprime to 10, as claimed.

On the other hand, for such a_1, we have $a_k = 12^{M+1-k} \cdot 13^{k-1} \cdot 6N$ for $1 \leqslant k \leqslant M+1$, and $a_{M+1} = a_{M+2} = \cdots = 6 \cdot 13^M \cdot N$.

Comment. Another way to show that the sequence a_n is eventually constant is to define a function f such that $f(a_n)$ is decreasing. One such function is a totally multiplicative $f\colon \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$ defined on primes by

$$f(p) = p \text{ for prime } p \neq 13, 31, \quad f(13) = 12 - \epsilon, \ f(31) = 30 - \epsilon,$$

for small positive ϵ. One can analyse the ratio $f(a_{n+1})/f(a_n)$ and show it is never more than 1. The range of f is well-ordered, so $f(a_n)$ is eventually constant, as is a_n. One can then work backwards to find possible values for a_1.

Comment. An alternate version of this problem is to prove that the sequence is eventually constant, instead of finding all possible values of a_1.

N4. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Determine all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}$ such that for every $n \in \mathbb{Z}_{>0}$, and every positive divisor d of n, there exists a positive divisor e of n such that n divides $d + f(e)$.

(Croatia)

Answer: The only such function is $f(n) = -n$.

Solution 1. If $f(n) = -n$, then given $n \in \mathbb{Z}_{>0}$ and a divisor d of n we can take $e = d$. Then $d + f(e) = d + f(d) = 0$ is divisible by n, so $f(n) = -n$ is indeed a solution.

In what follows, when we refer to divisors we are only considering positive divisors. Given a function f satisfying the conditions of the problem let S_n be the set of pairs (d, e) of divisors of n such that $n \mid d + f(e)$.

Claim 1. For every divisor e of n, there exists at most one divisor d of n such that $(d, e) \in S_n$.

Proof. Suppose d_1 and d_2 are such that $(d_1, e) \in S_n$ and $(d_2, e) \in S_n$. This means $n \mid d_1 + f(e)$ and $n \mid d_2 + f(e)$. Subtracting gives $n \mid d_1 - d_2$. As $1 \leqslant d_1, d_2 \leqslant n$, we get $d_1 = d_2$. □

The condition of the problem says that for every divisor d of n, there exists a divisor e of n with $(d, e) \in S_n$. As the number of divisors of n is finite, this fact, together with Claim 1, shows that S_n is the graph of a bijection from the set of divisors of n to itself.

Given a positive integer n and a divisor e of n, we denote the unique d such that $(d, e) \in S_n$ by $X_n(e)$.

Claim 2. The function f is injective.

Proof. If $f(a) = f(b)$, then $X_{ab}(a) = X_{ab}(b)$ contradicting injectivity of X_{ab}. □

We will prove by strong induction on the number of prime divisors of n (counted with multiplicity) that $f(n) = -n$.

Suppose we have proved the inductive hypothesis for all values of n with fewer than k prime divisors for some nonnegative integer k. If m has k prime divisors, we will prove that $f(m) = -m$.

Set $n = m$ in the condition of the question. Our inductive hypothesis implies that $f(e) + e = 0$ for all proper divisors e of m, so $X_m(e) = e$. As X_m is a bijection, $X_m(m) = m$. Hence $m \mid f(m)$.

Lemma. If m has k prime divisors and $f(m) \neq 0$ then $f(m) = -m$.

Proof. Let p be a prime number coprime to $f(m)$. Let $d = X_{mp}(m)$. If $p \mid d$, then $p \mid f(m)$, a contradiction. Hence d is a divisor of m. By the inductive hypothesis $X_{mp}(e) = e$ for all proper divisors e of m. As X_{mp} is a bijection, this implies $X_{mp}(m) = m$. Therefore $p \mid f(m) + m$. This is true for infinitely many primes p, so $f(m) = -m$ in this case. □

Thus we have reduced to considering the case where $f(m) = 0$. Let p be a prime not dividing m. If d is any proper divisor of mp not equal to m, by the lemma, either $f(d) = -d$ or $f(d) = 0$. Since f is injective, $f(d) = -d$ for all such divisors of mp. This determines X_{mp} completely, and in particular $X_{mp}(mp) = m$. So $mp \mid f(mp) + m$.

Since $f(mp) \neq 0$, we can let q be a prime coprime to $f(mp)$. Consider $X_{mpq}(mp)$. As q does not divide $f(mp)$, $X_{mpq}(mp)$ cannot be divisible by q. Hence $X_{mpq}(mp)$ is divisor of mp. The lemma above applies to all divisors $d \neq mp$ of mp. That is, $f(d) = -d$ or $f(d) = 0$. As $f(m) = 0$ and f is injective, for all divisors $d \neq m, mp$ of mp, we have $f(d) = -d$ and $X_{mpq}(d) = d$. Therefore, $X_{mpq}(mp)$ is either m or mp.

As mp divides $f(mp) + m$, mp does not divide $f(mp) + mp$. Hence mpq does not divide $f(mp) + mp$. Therefore $X_{mpq}(mp) = m$ and q divides $f(mp) + m$. As this is true for infinitely many primes q, we have $f(mp) = -m$. This contradicts the injectivity of f as it holds for infinitely many primes p.

This completes the inductive step of the proof, showing that $f(m) = -m$. We have now shown that $f(n) = -n$ for all n by the principle of mathematical induction.

Solution 2. Proceed as in Solution 1 until the third last paragraph. We are assuming for contradiction that $f(m) = 0$ and $f(d) = -d$ for all positive integers d with fewer prime divisors than m. We have already deduced that if p is a prime not dividing m then $mp \mid f(mp) + m$.

Consider $mp^k \mid f(mp) + X_{mp^k}(mp)$ for a positive integer k. Since $p \mid f(mp) + m$, it follows that $X_{mp^k}(mp)$ is a divisor of mp^k which is not divisible by p. So it is a divisor of m. But $X_{mp^k}(d) = d$ for all proper divisors d of m. Hence $X_{mp^k}(mp) = m$ and $mp^k \mid f(mp) + m$. This is true for all k, so $f(mp) = -m$. This is true for infinitely many primes p, contradicting injectivity.

This completes the inductive step. Therefore $f(n) = -n$ for all n.

Solution 3. Proceed as in Solution 1 until the end of Claim 2.

Lemma. For every m, either $f(m) = 0$ or $f(m) = -d$ for some divisor $d \mid m$.

Proof. Suppose for contradiction that $f(m)$ is not of one of these forms. Let $p > m + |f(m)|$ be a prime and let $d \mid pm$ such that $mp \mid d + f(m)$.

- If $p \nmid d$ then $d \mid m$. So $|d + f(m)| \leqslant m + |f(m)| < p$, and $d + f(m) \neq 0$ by assumption. Hence $d + f(m)$ cannot be divisible by p.
- If $p \mid d$ then $p \mid f(m)$, a contradiction unless $f(m) = 0$.

In both cases we have a contradiction. □

We now show that $f(n) = -n$ for all n. Suppose for contradiction that n is minimal with $f(n) \neq -n$. So $f(d) = -d$ for all proper divisors $d \mid n$. Hence $f(n) = 0$.

Let p be a prime such that $p \nmid n$. Then $pn \mid f(n) + pn$ since $f(n) = 0$. Let $m \mid pn$ such that $pn \mid n + f(m)$. Note that $n \neq m$ and $n \mid f(m)$. If d is a proper divisor of n then $n \nmid f(d) = -d$. So $m \nmid n$, which combined with $m \mid pn$ gives $p \mid m$.

By the lemma, $n - pn \leqslant n + f(m) \leqslant n$. So $n + f(m)$ is only divisible by pn if $f(m) = -n$. By injectivity, m is independent of the prime p. But this implies that m is divisible by infinitely many primes, which is a contradiction.

Therefore $f(n) = -n$ for all n.

N5. The sequence of triples of positive integers $(a_0, b_0, c_0), (a_1, b_1, c_1), \ldots, (a_{2025}, b_{2025}, c_{2025})$ satisfies

$$(a_{i+1}, b_{i+1}, c_{i+1}) = \big(a_i + \gcd(b_i, c_i), b_i + \gcd(c_i, a_i), c_i + \gcd(a_i, b_i)\big)$$

for $0 \leqslant i \leqslant 2024$.

Determine the smallest possible value of a_{2025}.

Here $\gcd(x, y)$ *denotes the greatest common divisor of integers* x *and* y.

(India)

Answer: The smallest possible value of a_{2025} is $3 \cdot 2^{2025/3} = 3 \cdot 2^{675}$.

Solution. To see that $a_{2025} = 3 \cdot 2^{675}$ is possible, let $(a_0, b_0, c_0) = (3, 5, 6)$. A straightforward induction then shows that, for any integer $0 \leqslant k \leqslant 674$,

$$\begin{aligned}(a_{3k}, b_{3k}, c_{3k}) &= \big(3 \cdot 2^k,\ 5 \cdot 2^k,\ 6 \cdot 2^k\big)\\ (a_{3k+1}, b_{3k+1}, c_{3k+1}) &= \big(4 \cdot 2^k,\ 8 \cdot 2^k,\ 7 \cdot 2^k\big)\\ (a_{3k+2}, b_{3k+2}, c_{3k+2}) &= \big(5 \cdot 2^k,\ 9 \cdot 2^k,\ 11 \cdot 2^k\big),\end{aligned}$$

so that $a_{2025} = 3 \cdot 2^{675}$. We must show that any starting triple (a_0, b_0, c_0) yields $a_{2025} \geqslant 3 \cdot 2^{675}$.

For positive integers a, b, c, let

$$f(a, b, c) = a + \gcd(b, c) \quad \text{and} \quad F(a, b, c) = \big(f(a, b, c), f(b, c, a), f(c, a, b)\big),$$

so $(a_{i+1}, b_{i+1}, c_{i+1}) = F(a_i, b_i, c_i)$ for all $0 \leqslant i \leqslant 2024$.

For any positive integers b, c, k we have $\gcd(kb, kc) = k \gcd(b, c)$. Hence for any positive integers a, b, c, k we have $f(ka, kb, kc) = kf(a, b, c)$ and $F(ka, kb, kc) = kF(a, b, c)$.

We call a triple of positive integers *m-even* if it contains precisely m even numbers. Since $F(2a, 2b, 2c) = 2F(a, b, c)$, we observe that if (a, b, c) is 3-even then $F(a, b, c)$ is also 3-even. One can also straightforwardly check that:

- if (a, b, c) is 0-even then $F(a, b, c)$ is 3-even;
- if (a, b, c) is 1-even then $F(a, b, c)$ is 2-even;
- if (a, b, c) is 2-even then $F(a, b, c)$ is 0-even.

Thus for any positive integers a, b, c, we have $F^{(3)}(a, b, c)$ is 3-even.

Let w_i be the largest integer n such that $2^n \mid \gcd(a_i, b_i, c_i)$, i.e. $w_i = \nu_2(\gcd(a_i, b_i, c_i))$. Note that, in particular, $\gcd(a_i, b_i, c_i) \geqslant 2^{w_i}$.

Let $(a_i, b_i, c_i) = 2^{w_i}(a_i', b_i', c_i')$ where a_i', b_i', c_i' are positive integers. It now follows that $(a_{i+1}, b_{i+1}, c_{i+1}) = 2^{w_i} F(a_i', b_i', c_i')$, so $w_{i+1} \geqslant w_i$ for all i.

Moreover, $w_{i+3} > w_i$, since $(a_{i+3}, b_{i+3}, c_{i+3}) = 2^{w_i} F^{(3)}(a_i', b_i', c_i')$, and $F^{(3)}(a_i', b_i', c_i')$ is 3-even.

Since $w_{i+1} \geqslant w_i$ and $w_{i+3} > w_i$, from $w_0 \geqslant 0$ we find that w_0, w_1, w_2 are all at least 0, then w_3, w_4, w_5 are all at least 1, and in general we observe that $w_{3k}, w_{3k+1}, w_{3k+2}$ are all at least k.

We then have

$$\begin{aligned}a_{2025} &= a_0 + \sum_{i=0}^{2024} \gcd(b_i, c_i) \geqslant a_0 + \sum_{i=0}^{2024} \gcd(a_i, b_i, c_i)\\ &\geqslant a_0 + \sum_{i=0}^{2024} 2^{w_i} = a_0 + \sum_{k=0}^{674} \left(2^{w_{3k}} + 2^{w_{3k+1}} + 2^{w_{3k+2}}\right)\\ &\geqslant a_0 + \sum_{k=0}^{674} 3 \cdot 2^k = a_0 + 3\left(2^{675} - 1\right).\end{aligned}$$

As $2^{w_{2025}} \mid a_{2025}$ and $w_{2025} \geqslant 675$, then $2^{675} \mid a_{2025}$. Combining this with the above inequality, we obtain $a_{2025} \geqslant 3 \cdot 2^{675}$ as claimed.

Comment. The lower bound can help motivate finding the construction. We want $a_0 = 3$ to be the smallest entry, with exactly one even number among b_0 and c_0. Starting with $(3, 4, 9)$ also works.

N6. For positive integers n and k, let $f_k(n)$ denote the remainder when n is divided by $2^k - 1$. A positive integer n is *grouse* if

$$f_1(n) \leqslant f_2(n) \leqslant f_3(n) \leqslant f_4(n) \leqslant \cdots .$$

(a) Prove that there are infinitely many grouse numbers.

(b) Prove that there exists a positive integer M such that there are at most $M/2025^{2025}$ grouse numbers less than M.

(U.S.A.)

Solution for part (a). It suffices to show that for any positive integer k, we can find a grouse number which is at least $2^k - 1$.

Given positive integers k and m, we iteratively construct a sequence $x_k, x_{k-1}, \ldots, x_1$ as follows. Let $x_k = (2^k - 1)m$, and for $i < k$, let x_i be the smallest multiple of $2^i - 1$ such that $x_i \geqslant x_{i+1}$.

Lemma. Starting from any positive integers k and m, the resulting x_1 satisfies

$$(2^k - 1)m \leqslant x_1 < (2^k - 1)(m + 1).$$

Proof. The lower bound follows from $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k = (2^k - 1)m$. For the upper bound, note that $x_i \leqslant x_{i+1} + (2^i - 2)$ for each $1 \leqslant i \leqslant k - 1$, so

$$x_1 \leqslant x_k + \sum_{i=1}^{k-1} (2^i - 2) = x_k + 2^k - 2k < (2^k - 1)(m + 1). \qquad \square$$

The lemma also applies starting from any x_i for $1 \leqslant i \leqslant k$, that is, with k replaced by i, and m replaced by m' such that $x_i = (2^i - 1)m'$. We then obtain the same $x_i, x_{i-1}, \ldots, x_1$, and conclude that $(2^i - 1)m' \leqslant x_1 < (2^i - 1)(m' + 1)$.

We claim that, starting from any positive integer k with $m = 1$ (that is, starting with $x_k = 2^k - 1$), the resulting x_1 is grouse. By the lemma, $f_k(x_1) = x_1 - (2^k - 1) = x_1 - x_k$. Applying the lemma starting from x_i, we have $f_i(x_1) = x_1 - (2^i - 1)m' = x_1 - x_i$. Since $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k$ this implies

$$f_1(x_1) \leqslant f_2(x_1) \leqslant f_3(x_1) \leqslant \cdots \leqslant f_k(x_1).$$

Now for all $\ell > k$, we show that $f_\ell(x_1) = x_1$, implying x_1 is grouse. Since $m = 1$, the lemma gives $x_1 < (2^k - 1)(m + 1) = 2(2^k - 1)$, and $2(2^k - 1) < 2^{k+1} - 1 \leqslant 2^\ell - 1$. As $x_1 < 2^\ell - 1$ then $f_\ell(x_1) = x_1$. Hence x_1 is grouse, as claimed.

For this grouse x_1 we have $x_1 \geqslant x_k = 2^k - 1$. So for any positive integer k we have found a grouse number which is at least $2^k - 1$, as desired. $\square$

Solution 1 for part (b). We first prove the following claim.

Claim. Let k be a positive integer, and suppose $n \in [2^k, 2^{k+1})$ is grouse. Then $n \geqslant 2^{k+1} - 3 \cdot 2^{k/2}$.

Proof. Let $x_k = 2^k - 1$, and consider $x_k, x_{k-1}, \ldots, x_1$ as constructed in part (a). As $2^k - 1 < n < 2^{k+1}$ we have $f_k(n) \leqslant n - (2^k - 1) = n - x_k$, that is, $x_k \leqslant n - f_k(n)$. We now use downwards induction on i to show that for each $1 \leqslant i \leqslant k$, we have

$$x_i \leqslant n - f_i(n). \tag{1}$$

We have demonstrated the case $i = k$; now consider $i < k$ and suppose $x_{i+1} \leqslant n - f_{i+1}(n)$. The number $n - f_i(n)$ is a multiple of $2^i - 1$ and, as n is grouse, $n - f_i(n) \geqslant n - f_{i+1}(n) \geqslant x_{i+1}$. By definition of x_i then $n - f_i(n) \geqslant x_i$, completing the induction.

On the other hand, we claim that

$$x_i = 2^{k+1} - 2^i - 2^{k+1-i} + 1 = (2^i - 1)(2^{k+1-i} - 1) \tag{2}$$

for each positive integer i such that $k/2 \leqslant i \leqslant k$. This is clear when $i = k$; we proceed again by downward induction, considering $i < k$ and supposing $x_{i+1} = (2^{i+1} - 1)(2^{k-i} - 1)$. Then

$$(2^i - 1)(2^{k+1-i} - 1) - x_{i+1} = (2^i - 1)(2^{k+1-i} - 1) - (2^{i+1} - 1)(2^{k-i} - 1) = 2^i - 2^{k-i}.$$

As $k/2 \leqslant i < k$ then $0 \leqslant 2^i - 2^{k-i} < 2^i - 1$. So $(2^i - 1)(2^{k+1-i} - 1)$ is the smallest multiple of $2^i - 1$ which is at least x_{i+1}. Combining (1) and (2) and letting $i = \lceil k/2 \rceil$ then yields

$$n \geqslant n - f_{\lceil k/2 \rceil}(n) \geqslant x_{\lceil k/2 \rceil} = 2^{k+1} - 2^{\lceil k/2 \rceil} - 2^{k+1-\lceil k/2 \rceil} + 1,$$

and it remains to show that the final quantity is at least $2^{k+1} - 3 \cdot 2^{k/2}$, or equivalently,

$$2^{\lceil k/2 \rceil} + 2^{k+1-\lceil k/2 \rceil} \leqslant 3 \cdot 2^{k/2} + 1.$$

When k is even, say $k = 2\ell$, we have $2^{\lceil k/2 \rceil} + 2^{k+1-\lceil k/2 \rceil} = 3 \cdot 2^\ell = 3 \cdot 2^{k/2}$. When k is odd, say $k = 2\ell - 1$, we have $2^{\lceil k/2 \rceil} + 2^{k+1-\lceil k/2 \rceil} = 2^{\ell+1} = 2\sqrt{2} \cdot 2^{k/2}$. The claim then follows. □

The claim shows that the number of grouse numbers in $[2^k, 2^{k+1})$ is bounded above by $3 \cdot 2^{k/2}$. Let $M = 2^{b+1}$ for an integer b to be determined later. Then the number of grouse numbers less than M is bounded above by

$$3 \sum_{k=0}^{b} 2^{k/2} = \frac{3}{\sqrt{2} - 1} \left(2^{\frac{b+1}{2}} - 1 \right) < \frac{3}{\sqrt{2} - 1} 2^{\frac{b+1}{2}}.$$

The last expression is smaller than $M/2025^{2025}$ if and only if

$$\frac{3 \cdot 2025^{2025}}{\sqrt{2} - 1} < 2^{\frac{b+1}{2}}.$$

The base of the exponent on the right hand side is larger than 1, so this is true for sufficiently large b, completing the proof of (b).

Solution 2 for part (b). Let us call a residue class r modulo d *cooked* if $n \equiv r \pmod{d}$ implies that n is not grouse. It suffices to find an M such that at most $2025^{-2025} M$ residue classes modulo M are not cooked.

Lemma. For any positive integer k, there are at least $\binom{2^{2k-1}-1}{2}$ residue classes modulo $(2^{2k-1} - 1)(2^{2k+1} - 1)$ which are cooked.

Proof. Let i, j be integers such that $0 \leqslant j < i \leqslant 2^{2k-1} - 2$. By definition, if n is such that $f_{2k-1}(n) = i$ and $f_{2k+1}(n) = j$, then n is not grouse. Using the well known fact that $\gcd(2^a - 1, 2^b - 1) = 2^{\gcd(a,b)} - 1$, we know that $\gcd(2^{2k-1} - 1, 2^{2k+1} - 1) = 2^{\gcd(2k-1, 2k+1)} - 1 = 1$. As $2^{2k-1} - 1$ and $2^{2k+1} - 1$ are coprime, we can apply the Chinese Remainder Theorem to show that for any pair of integers (i, j) there is a unique residue class r mod $(2^{2k-1} - 1)(2^{2k+1} - 1)$ such that $r \equiv i \pmod{2^{2k-1} - 1}$ and $r \equiv j \pmod{2^{2k+1} - 1}$. Therefore for each pair (i, j) such that $1 \leqslant j < i \leqslant 2^{2k-1} - 2$, there is a corresponding residue class r mod $(2^{2k-1} - 1)(2^{2k+1} - 1)$ which is cooked. There are $\binom{2^{2k-1}-1}{2}$ such combinations for i and j, and the corresponding $\binom{2^{2k-1}-1}{2}$ residue classes mod $(2^{2k-1} - 1)(2^{2k+1} - 1)$ are distinct and cooked. □

If the integer n is grouse, it is not in a cooked residue class for any modulus d. Therefore, the Chinese Remainder Theorem implies that if there are a_i residue classes modulo d_i which are not cooked for $1 \leqslant i \leqslant \ell$, and if the d_i are all pairwise coprime, then we have that at most $\prod_{i=1}^{\ell} a_i$ residue classes modulo $M = \prod_{i=1}^{\ell} d_i$ are not cooked. Phrased differently, if the fraction of residue classes modulo d_i that are not cooked is a_i/d_i, then the fraction of residue classes modulo M are not cooked is at most $\prod_{i=1}^{\ell} a_i/d_i$.

Using this, we find a suitable set of d_i which satisfy the bound. Define a sequence recursively by $k_1 = 2$ and

$$k_j = \prod_{i=1}^{j-1}(2k_i - 1)(2k_i + 1).$$

Let

$$M_l = \prod_{i=1}^{\ell}(2^{2k_i-1} - 1)(2^{2k_i+1} - 1).$$

We will show this satisfies the conditions for (b) once ℓ is chosen sufficiently large so that $(\frac{10}{11})^{\ell} < 2025^{-2025}$.

It is easy to see that $2k_1 - 1, 2k_1 + 1, 2k_2 - 1, 2k_2 + 1, \ldots, 2k_\ell - 1, 2k_\ell + 1$ are pairwise coprime. Hence, again using the fact that $\gcd(2^a - 1, 2^b - 1) = \gcd(2^{\gcd(a,b)} - 1)$, we see that $(2^{2k_i-1} - 1)(2^{2k_i+1} - 1)$ are pairwise coprime for $1 \leqslant i \leqslant \ell$. Furthermore, for $k \geqslant 2$,

$$\frac{2^{2k-2} - 1}{2^{2k+1} - 1} = \frac{1}{8} - \frac{7}{8(2^{2k+1} - 1)} \geqslant \frac{3}{31} > \frac{1}{11}.$$

Hence the fraction of residue classes modulo M which are not cooked is at most

$$\prod_{i=1}^{\ell}\left(1 - \frac{\binom{2^{2k_i-1}-1}{2}}{(2^{2k_i-1} - 1)(2^{2k_i+1} - 1)}\right) = \prod_{i=1}^{\ell}\left(1 - \frac{2^{2k_i-2} - 1}{2^{2k_i+1} - 1}\right) < \left(\frac{10}{11}\right)^{\ell} < 2025^{-2025},$$

and we obtain the desired bound.

N7. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. A function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is said to be *bonza* if

$$f(a) \text{ divides } b^a - f(b)^{f(a)}$$

for all positive integers a and b.

Determine the smallest constant c such that $f(n) \leqslant cn$ for any bonza function f and any positive integer n.

(Colombia)

Answer: The smallest such constant is $c = 4$.

Solution. We first show that $c = 4$ is a valid bound. Note that $f(n) = n$ is a solution to the functional equation and trivially satisfies $f(n) \leqslant 4n$ for all n. We will now assume throughout that there exists an integer m such that $f(m) \neq m$.

Substitute $b = a$ into the given condition. Then $f(a) \mid a^a$ for all a. In particular, if p is prime, we see that $f(p)$ is a power of p.

Claim. If p is prime and $f(n) \neq n$, then $f(p) = 1$ or $p \mid n - f(n)$.

Proof. Substitute $b = n$ and $a = p$ in the statement. If $f(p) \neq 1$, then as $f(p)$ is a power of p, we have $p \mid f(p)$ and hence

$$p \mid n^p - f(n)^{f(p)}.$$

By Fermat's Little Theorem, $n^p \equiv n \pmod p$. As $f(p)$ is a power of p, we also have $f(n)^{f(p)} \equiv f(n) \pmod p$. Therefore $p \mid n - f(n)$. □

Now if q is a prime such that $q > |m - f(m)|$, then using the claim with $p = q$ and $n = m$, since $q \nmid m - f(m)$, we have $f(q) = 1$. In other words, for sufficiently large primes q, $f(q) = 1$.

We now claim that for any odd prime p, $f(p) = 1$. By the previous paragraph, any sufficiently large prime q satisifes $f(q) = 1$. For such a q we have $f(q) \neq q$, so using the claim with $n = q$, we have $f(p) = 1$ or $f(q) \equiv q \equiv 1 \pmod p$. If we additionally have $q \not\equiv 1 \pmod p$, then we can conclude $f(p) = 1$. But there exist infinitely many primes q such that $q \not\equiv 1 \pmod p$: this follows immediately from Dirichlet's Theorem on primes in arithmetic progressions, or can be proven directly (see comments). Taking q to be such a prime (that is, with $q > |m - f(m)|$ and $q \equiv 1 \pmod p$), we conclude $f(p) = 1$.

For any $n \in \mathbb{Z}_{>0}$, we have $f(n) \mid n^n$, so every prime divisor of $f(n)$ also divides n. However, if q is an odd prime divisor of n, substituting $a = n$, $b = q$ and $f(q) = 1$ in the problem statement yields

$$f(n) \mid q^n - 1,$$

so q is not a divisor of $f(n)$. Hence the only prime divisor of $f(n)$ is 2. So for all $n \in \mathbb{Z}_{>0}$, $f(n)$ is a power of 2. Moreover, if n is odd then $f(n) = 1$.

Let $a = 2^x y$ where $x \geqslant 0$ and y is odd. Then $f(a) = 2^z$ for some $z \geqslant 0$. We claim $z \leqslant x + 2$, which implies $f(a) \leqslant 4a$ as desired. For any odd b we have $f(b) = 1$, and the problem statement yields

$$2^z \mid b^{2^x y} - 1, \quad \text{i.e.} \quad b^{2^x y} \equiv 1 \pmod{2^z}.$$

Now $\phi(2^z)$ is a power of 2, and y is odd, so there exists a positive integer c such that $yc \equiv 1 \pmod{\phi(2^z)}$. Letting $b = 5^c$, we obtain, by the Fermat–Euler Theorem,

$$1 \equiv b^{2^x y} \equiv (5^{yc})^{2^x} \equiv 5^{2^x} \pmod{2^z}.$$

But $5^{2^x} - 1 = \left(5^{2^{x-1}} + 1\right)\left(5^{2^{x-1}} - 1\right)$, and repeatedly factorising differences of squares yields

$$5^{2^x} - 1 = \left(5^{2^{x-1}} + 1\right)\left(5^{2^{x-2}} + 1\right) \cdots \left(5^2 + 1\right)(5 + 1)(5 - 1).$$

As each $5^{2^k}+1 \equiv 2 \pmod 4$, we have $\nu_2\left(5^{2^k}+1\right)=1$ and hence $\nu_2\left(5^{2^x}-1\right)=x+2$. Since $2^z \mid 5^{2^x}-1$ we obtain $z \leqslant x+2$. Hence for all $n \in \mathbb{Z}_{>0}$, $f(n) \leqslant 4n$, as desired.

We now show that no constant better than $c=4$ is possible. Define $f(n)$ by

$$f(n)=\begin{cases}1 & \text{if } n \text{ is odd,}\\ 2 & \text{if } n \text{ is even and } n \neq 4,\\ 16 & \text{if } n=4.\end{cases}$$

We verify the condition on f directly, for all values of a.

- If a is odd then $f(a)=1$ and the condition $f(a) \mid b^a - f(b)^{f(a)}$ is immediate.
- If a is even and $a \neq 4$, then $f(a)=2$, and the condition becomes $2 \mid b^a - f(b)^2$. By construction of f, b and $f(b)$ have the same parity, so the condition holds.
- If $a=4$, then $f(a)=16$ and the condition becomes $16 \mid b^4 - f(b)^{16}$. If b is even then $f(b)$ is also even, so b^4 and $f(b)^{16}$ are both divisible by 16. If b is odd then $b^4-f(b)^{16}=b^4-1=(b-1)(b+1)(b^2+1)$. The three factors are all even, and $b-1, b+1$ differ by 2, so one of them is divisible by 4. Hence b^4-1 is divisble by 16, as required.

Thus this function satisfies the condition, and as $f(4)=16$, the value $c=4$ is optimal.

Comment. The solutions to the functional equation are $f(n)=n$ for all n; $f(n)=1$ for all n; and

$$f(n)=\begin{cases}1 & \text{if } n \text{ is odd,}\\ 2^{g(n)} & \text{if } n \text{ is even,}\end{cases}$$

where $1 \leqslant g(2) \leqslant 2$ and $1 \leqslant g(n) \leqslant v_2(n)+2$ for $n \geqslant 2$.

Comment. We can prove that for any odd prime p there exist infinitely many primes $q \not\equiv 1 \pmod p$ without using Dirichlet's Theorem.

Assume for contradiction that there are finitely many such primes q. Let N be the product of all of them. Consider the number $pN-1$. Note that $\gcd(pN-1,q)=1$ for all primes q such that $q \not\equiv 1 \pmod p$. So all prime factors of $pN-1$ are congruent to $1 \pmod p$. But this implies that $pN-1 \equiv 1 \pmod p$. So $2 \equiv 0 \pmod p$, a contradiction.

Comment. The statement $\nu_2\left(5^{2^x}-1\right)=x+2$ can also be proved by lifting the exponent:

$$\nu_2(5^{2^x}-1)=\nu_2(5-1)+\nu_2(5+1)+\nu_2(2^x)-1=2+1+x-1=x+2.$$

N8. Prove that there are finitely many prime numbers p such that

- $p-8$ is a perfect square, and
- for all odd positive integers $k < \sqrt{p}$, the number $p-k^2$ has at most two distinct prime factors.

(Romania)

Solution. Let $p = n^2+8$ be such a prime. So p is odd and $p \geqslant 8$; indeed $p \geqslant 11$, and n is odd. Then $3 < \sqrt{p}$, and setting $k=3$ we observe $p-9 = (n+1)(n-1)$ is the product of at most two distinct primes. Since p is odd, one of these primes is 2. So $(n+1)(n-1) = 2^a q^b$ for some odd prime q and nonnegative integers a and b.

Since n is odd, $\gcd(n-1, n+1) = 2$. So $\{n-1, n+1\} = \{2^{a-1}, 2q^b\}$ or $\{2, 2^{a-1}q^b\}$. The latter case implies $n=3$, which gives the solution $p=17$. Hence $n-1$, $n+1$ coincide with 2^{a-1}, $2q^b$ in some order. We consider the two cases separately.

Case 1: $n = 2^{a-1}+1 = 2q^b - 1$.

If $b=0$ then $n=1$ and $p=9$, which is not prime, so we may assume $b \geqslant 1$. Then $2^{a-2} = q^b - 1 = (q-1)(1+q+\cdots+q^{b-1})$. So $1+q+\cdots+q^{b-1}$ is a power of 2, and as q is odd, it has the same parity as b. Since the only odd power of 2 is 1, if b is odd then $b=1$. Thus $b=1$ or b is even. If b is even, $2^{a-2} = (q^{b/2}+1)(q^{b/2}-1)$. So $q^{b/2}+1$ and $q^{b/2}-1$ are powers of 2 which differ by 2. Hence $q^{b/2}=3$, so $n=17$ and $p=297$, which is not prime. So we may assume $b=1$.

From $b=1$ we have $2^{a-1}+1 = 2q-1$, so $q = 2^{a-2}+1$. If $a=2$ then $n=3$ and $p=17$, which is a solution. Hence we may assume $a \geqslant 3$ and q is a Fermat prime (an odd prime one more than a power of 2). If an odd integer $d>1$ is a divisor of $a-2$, then letting $a-2=de$ we obtain $q = 2^{de}+1 = (2^e+1)(2^{e(d-1)} - 2^{e(d-2)} + \cdots - 2^e + 1)$, contradicting primality of q. So $a = 2^c+2$ for some nonnegative integer c, and $q = 2^{2^c}+1$. If $c \geqslant 4$ then $2^{a-2} \equiv 2^{2^c} \equiv (2^{16})^{2^{c-4}} \equiv 1^{2^{c-4}} \equiv 1 \pmod{17}$ by Fermat's Little Theorem. So $n \equiv 2^{a-1}+1 \equiv 3$ and $p \equiv 0 \pmod{17}$. Hence $p=17$. Otherwise $c<4$, leading to only 4 possibilities for a, and hence for n and p, including the solutions $p=89$ and 1097.

Case 2: $n = 2^{a-1}-1 = 2q^b+1$.

If a is even then $q^b = 2^{a-2}-1 = (2^{(a/2-1)}+1)(2^{(a/2-1)}-1)$. So two powers of q differ by 2. These must be 1 and 3, which does not lead to a solution. Hence we may assume a is odd and $a \geqslant 3$. (We could prove that q is a Mersenne prime similarly to the previous case, but it is not necessary.)

Lemma. For all but finitely many n, there exists a prime divisor r of n such that $r \leqslant \sqrt{n}$ and p is a quadratic residue modulo r.

(Note that as n is odd, the divisor r is also odd. And since $p = n^2+8$, the lemma applies for all but finitely many p.)

Proof. If $r \mid n$ then $p = n^2+8 \equiv 8 \pmod{r}$. So 2 is a quadratic residue modulo r if and only if $2 \cdot 2^2 \equiv 8 \equiv p$ is a quadratic residue modulo r.

Case 2a: $a-1$ *is not a power of 2.*

Let $s>1$ be an odd divisor of $a-1$. As $a-1$ is even, s is a proper divisor of $a-1$. Then 2^s-1 divides $2^{a-1}-1 = n$ and $2^s-1 \leqslant \sqrt{n}$. Let r be a prime divisor of 2^s-1. Since $2^{s+1} \equiv 2 \pmod{r}$ and $s+1$ is even, 2 is a quadratic residue modulo r, hence so too is p.

Case 2b: $a-1$ *is a power of 2.*

Let $a-1=2^d$ for some nonnegative integer d. If $d \geqslant 4$ then $n=2^{a-1}-1 \equiv 2^{2^d}-1 \equiv 0 \pmod{17}$ by Fermat's Little Theorem. So $17 \mid n$, and $6^2 \equiv 2 \pmod{17}$, so 2 (hence also p) is a quadratic residue modulo 17. For all but finitely many n we have $17 \leqslant \sqrt{n}$ and $d \geqslant 4$. For such n we may take $r=17$. □

Excluding the finitely many n from the lemma, we thus obtain an r as claimed. There then exists a positive integer $m<r$ such that $m^2 \equiv p \pmod{r}$. Since r is odd, replacing m with $r-m$ if necessary, we may assume m is odd.

Again excluding finitely many n (hence p), we may assume $n \geqslant 25$. Using $n \geqslant 25$ and $n \geqslant r^2$ then $p=n^2+8>n^2 \geqslant 25r^2$. As $m<r$ then $p>(5r)^2 \geqslant (4r+m)^2$. Then $p-m^2, p-(4r-m)^2, p-(4r+m)^2$ are divisible by 2, r and no other primes. Considering greatest common divisors, we have:

1. $\gcd(p-m^2, p-(4r-m)^2) = \gcd(p-m^2, (4r-m)^2-m^2) \mid 8r(2r-m)$,
2. $\gcd(p-m^2, p-(4r+m)^2) = \gcd(p-m^2, (4r+m)^2-m^2) \mid 8r(2r+m)$,
3. $\gcd(p-(4r+m)^2, p-(4r-m)^2) = \gcd(p-(4r+m)^2, (4r+m)^2-(4r-m)^2) \mid 16rm$.

Since r and m are odd, the highest power of 2 that divides any of these gcds is 16, and the highest power of r that divides any of the gcds is r. Therefore, at most one of $p-m^2$, $p-(4r-m)^2$, $p-(4r+m)^2$, is divisible by r^2, and at most one is divisible by 32. Therefore at least one of them must divide $16r$. Hence the smallest among them is at most $16r$, that is, $p-(4r+m)^2 \leqslant 16r$. As $m<r$ are both odd we have $r-2 \geqslant m$, so $5r-2 \geqslant 4r+m$, hence $p-(5r-2)^2 \leqslant p-(4r+m)^2 \leqslant 16r$. So $p \leqslant (5r-2)^2+16r<25r^2$, contradicting our previous bound $p \geqslant 25r^2$.

Therefore, there are only finitely many p satisfying the given conditions.

Comment. It can be shown, using similar arguments to those above, that the only primes p satisfying the given conditions are 17, 89, 233 and 1097. This requires numerous arithmetic verifications.

Comment. There are various ways to show that any prime $r \mid p-k^2$ must be relatively large. The crux of this problem is deriving a contradiction by finding a small prime which divides $p-k^2$ as in the lemma.

Comment. Since the two cases involve equalities $2^{a-2}=q^b-1$ and $2^{a-2}=q^b+1$, when $b>1$ they can be dealt with using Catalan's Conjecture.

IMO Problem Selection Committee

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