Fourteenth International Olympiad, 1972

1972/1.

Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.

1972/2.

Prove that if $n \geq 4$, every quadrilateral that can be inscribed in a circle can be dissected into n quadrilaterals each of which is inscribable in a circle.

1972/3.

Let m and n be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer. ($0! = 1$.)

1972/4.

Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities

$$\begin{array}{rcl}
(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) & \leq & 0 \\
(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) & \leq & 0 \\
(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) & \leq & 0 \\
(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) & \leq & 0 \\
(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) & \leq & 0
\end{array}$$

where x_1, x_2, x_3, x_4, x_5 are positive real numbers.

1972/5.

Let f and g be real-valued functions defined for all real values of x and y, and satisfying the equation

$$f(x+y) + f(x-y) = 2f(x)g(y)$$

for all x, y. Prove that if $f(x)$ is not identically zero, and if $|f(x)| \leq 1$ for all x, then $|g(y)| \leq 1$ for all y.

1972/6.

Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.